PROCEEDINGS

OF THE

AMERICAN SOCIETY OF CIVIL ENGINEERS

VOL. 73 MARCH, 1947 No. 3

TECHNICAL PAPERS

AND

DISCUSSIONS

Published monthly, except July and August, at Prince and Lemon Streets, Lancaster, Pa., by the American Society of Civil Engineers. Editorial and General Offices at 33 West Thirty-Ninth Street, New York 18, N. Y. Reprints from this publication may be made on condition that the full title of Paper, name of Author, page reference, and date of publication by the Society are given.

Entered as Second-Class Matter, September 23, 1937, at the Post Office at Lancaster, Pa., under the Act of March 3, 1879. Acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, authorized on July 5, 1918.

Subscription (if entered before January 1) $8.00 per annum Price $1.00 per copy

Printed in the United States of America

Current Papers and Discussions

	Published	Discussion closes
Symposium: Uplift Pressure in and Beneath Dams Discussion in June, Sept., 1946, Mar., 1947	Dec., 1945	Closed*
Ripperger, E. A., and Davids, N. Critical Stresses in a Circular Ring Discussion in Oct., Nov., 1946	Feb., 1946	Closed*
Forbes, Hyde. Landslide Investigation and Correction Discussion in May, Sept., Nov., Dec., 1946, Mar., 1947	Feb., 1946	Closed*
Winter, George. Strength of Thin Steel Compression Flanges Discussion in June, Oct., Dec., 1946, Jan.,Mar., 1947	Feb., 1946	Closed*
Wood, Charles P. Factors Controlling the Location of Various Types of Industry Discussion in Nov., 1946, Mar., 1947	Mar., 1946	Closed*
Barnett, Joseph. Express Highway Planning in Metropolitan Areas Discussion in May, Sept., Nov., 1946, Jan. 1947	Mar., 1946	Closed*
Howson, Louis R. Future Costs and Their Effects on Engineering Budgets Discussion in Jan., 1947	Mar., 1946	Closed*
Sette, F. J. The Planning of Aerial Photographic Projects Discussion in Sept., Oct., 1946, Mar., 1947	Mar., 1946	Closed*
Dunham, John W. Design Live Loads in Buildings Discussion in Oct., 1946	Apr., 1946	Closed*
Peterson, F. G. Eric. Effect of Stress Distribution on Yield Points Discussion in Sept., Nov., Dec., 1946, Mar., 1947	Apr., 1946	Closed*
Baker, Donald M. Some Thoughts on Engineering Education Discussion in Sept., Oct., Dec., 1946, Feb., 1947	Apr., 1946	Closed*
Rose, Edwin. Thrust Exerted by Expanding Ice Sheet Discussion in Dec., 1946	May, 1946	Apr. 1, 1947
Dell, George H. Moment-Stiffness Relations in Continuous Frames with Prismatic Members Discussion in Oct., 1946	May, 1946	Apr. 1, 1947
Bermel, Karl J., and Sanks, Robert L. Model Study of Brown Canyon Debris Barrier Discussion in Nov., Dec., 1946	May, 1946	Apr. 1, 1947
Jacob, C. E. Effective Radius of Drawdown Test to Determine Artesian Well Discussion in Nov., Dec., 1946, Feb., 1947	May, 1946	Apr. 1, 1947
Report: Opportunities for Sanitary Engineers in the Postwar Period: Progress Report of the Committee on Advancement of Sanitary Engineering of the Sanitary Engineering Division	May, 1946	
Moore, R. L. Observations on the Behavior of Aluminum Alloy Test Girders	June, 1946	Apr. 1, 1947
Hansen, Howard J. Design of Plywood I-Beams Discussion in Nov., 1946, Jan., 1947	June, 1946	Apr. 1, 1947
Tommerup, Carl C. H. Rigid-Frame Structures Subject to Nonuniform Thermal Action Discussion in Nov., 1946, Jan., Mar., 1947	June, 1946	Apr. 1, 1947
Middlebrooks, T. A., and Jervis, William H. Relief Wells for Dams and Levees Discussion in Oct., Dec., 1946, Jan., Feb., 1947	June, 1946	Apr. 1, 1947
Report: Advances in Sewage Treatment and Present Status of the Art: Third Progress Report of the Committee of the Sanitary Engineering Division on Sewerage and Sewage Treatment	June, 1946	
Stevens, J. W. Truck Speed and Time Loss on Grades Discussion in Dec., 1946, Mar., 1947	Sept., 1946	May, 1, 1947
Underwood, P. H. Space Resection Problems in Photogrammetry Discussion in Dec., 1946, Jan., 1947	Sept., 1946	May 1, 1947
McGuinness, Charles L. Recharge and Depletion of Ground-Water Supplies Discussion in Mar., 1947	Sept., 1946	May 1, 1947
de Vries, Karl. Strength of Beams as Determined by Lateral Buckling Discussion in Dec., 1946, Feb., Mar., 1947	Sept., 1946	May 1, 1947
Ward, C. N., and Hunt, Henry J. Correction of Tailwater Erosion at Prairie Du Sac Dam	Oct., 1946	May 1, 1947
Benscoter, Stanley U. Matrix Analysis of Continuous Beams Discussion in Feb., 1947	Oct., 1946	May 1, 1947
Machis, Alfred. Experimental Observations on Grouting Sands and Gravels Discussion in Feb., Mar., 1947	Nov., 1946	May 1, 1947
Symposium: Cleaning and Grouting of Limestone Foundations, Tennessee Valley Authority Discussion in Mar., 1947	Dec., 1946	May 1, 1947
Matthes, Gerard A. Mississippi River Cutoffs Discussion in Mar., 1947	Jan., 1947	June 1, 1947
Tan, Ek-Khoo. Stability of Soil Slopes	Jan., 1947	June 1, 1947
Muldrow, W. C. Forecasting Productivity of Irrigable Lands	Feb., 1947	July 1, 1947
Freudenthal, Alfred M. Reflections on Standard Specifications for Structural Design	Feb., 1947	July 1, 1947
Blaisdell, Fred W. Development and Hydraulic Design, Saint Anthony Falls Stilling Basin	Feb., 1947	July 1, 1947
DuVal, Miles P. The Marine Operating Problems, Panama Canal, and the Solution	Feb., 1947	July 1, 1947
Claybourn, J. G. Sea Level Plan for Panama Canal	Feb., 1947	July 1, 1947

Note.—The closing dates herein published are final except when names of prospective discussers are registered for special extension of time.

* Publication of closing discussion pending.

VOL. 73 PROCEEDINGS NO. 3

CONTENTS FOR MARCH, 1947

PAPERS

REPORT

DISCUSSIONS

CONTENTS FOR MARCH, 1947 (*Continued*)

A list of "Current Papers and Discussions" may be found on the page preceding the table of contents

HINTS TO AUTHORS

Those who are planning papers for submission to PROCEEDINGS *will expedite Committee action measurably by first studying the standard instructions as to style, content, and format. For reprints, address the Manager, Technical Publications, 33 W. 39th St., New York 18, N. Y.*

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

THEORY OF INELASTIC BENDING WITH REFERENCE TO LIMIT DESIGN

By Alexander Hrennikoff,[1] Assoc. M. ASCE

Synopsis

For the most part this paper contains the description of a method for analyzing statically indeterminate flexural structures loaded beyond the elastic limit or structures comprising material that does not obey Hooke's law. The relations between the unit strains and the unit stresses (both normal and tangential), bending moments, angle changes, and deflections, produced under these circumstances, are derived. The effects of removing a load and of moving a load along the span are also considered. Furthermore, the method is applied to mild steel I-beams, and is illustrated by a number of examples bearing on the "Theory of Limit Design" presented by J. A. Van den Broek,[2] M. ASCE. Brief discussion of this theory together with observations concerning its validity, based on the results obtained, is included in Section 18.

1. Introduction

The method proposed in this paper entails the construction of several curves derived from the stress-strain curve of the material of the structure, making proper allowance for the shape of the cross section of its members. Once the requisite curves are constructed, the indeterminate moments are found by trial. The labor involved in such an analysis is not prohibitive, although it is greater than that involved in the usual elastic analysis. A new set of curves is needed if the material of the structure or the cross section of its members is different.

For the most part, this method, with its illustrative numerical solutions, is novel, although a simpler case of the same problem, involving a rectangular beam, has been treated, in a somewhat different way, by S. Timoshenko.[3]

The method is based on the assumption that the cross sections of the members remain plane throughout the entire range of flexure. The same as-

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by **August 1, 1947.**

[1] Associate Prof., Civ. Eng., Dept. of Civ. Eng., Univ. of British Columbia, Vancouver, B.C., Canada.

[2] "Theory of Limit Design," by J. A. Van den Broek, *Transactions*, ASCE, Vol. 105, 1940, p. 638.

[3] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York and London, 1936, p. 45.

sumption has also been used by Professor Timoshenko[3] and by A. Nádai;[4] and it has some experimental corroboration.[5] In certain restricted regions the sections undoubtedly become warped by shear, but on the whole the assumption is believed to be substantially true.

In addition to this assumption the theory is restricted by the following limitations:

1. The structure consists of prismatic members with I-beam, channel, or rectangular sections, symmetrical about their neutral axes;
2. The stress-strain curve is the same in tension and compression (the only limitation of the shape of this curve);
3. The bending moment diagram of the loaded structure is assumed to be bounded by straight lines (a curved diagram being approximated by a polygonal shape);
4. The effect of normal forces is ignored;
5. Deformations produced by shearing forces are disregarded;
6. Instability is not considered to be a factor; and
7. Deformations of the structure are assumed small.

Some additional assumptions, useful in expediting the work but otherwise avoidable, will be stated subsequently.

Notation.—The letter symbols in this paper are defined where they first appear, in the text or by illustration, and are assembled for convenience of reference, in the Appendix.

2. Unit Strain—Bending Moment Relation in a Rectangular Beam

The functional relationship between the unit stress, σ, and the unit strain, ϵ, for the material of the beam must be stated explicitly in one of the following forms: As an equation, $\sigma = f(\epsilon)$; as a graph, such as Fig. 1; or as a table of corresponding values of the two variables.

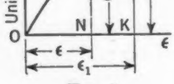


Fig. 1

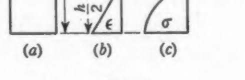


Fig. 2

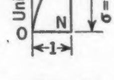


Fig. 3

Let a rectangular beam with the cross section shown in Fig. 2(*a*) be subjected to flexure. According to the stated assumptions, the unit strain diagram is linear (Fig. 2(*b*)), with zero strain at the center of the section and with equal strains ϵ at the outside fibers. It follows that the diagram of unit stress (Fig. 2(*c*)) consists of two equal parts resembling the stress-strain curve (Fig. 1) to point A, with the abscissa ϵ. The base $\overline{ON} = \epsilon$ in Fig. 1 is reduced in Fig. 2(*c*)

[4] "Plasticity," by A. Nádai, McGraw-Hill Book Co., Inc., New York, N. Y., 1931, p. 164.

[5] "Mechanics of Creep," by J. Marin, *Transactions*, ASCE, Vol. 108, 1943, p. 459.

to the length $\frac{1}{2}h$. When the strain in the outside fibers of the beam is increased to a value ϵ_1, in response to a greater bending moment, the stress curve in Fig. 2(*c*) will resemble a greater length of the stress-strain curve OMBK (Fig. 1), whose base ϵ_1 is again foreshortened to the same length, $\frac{1}{2}h$.

To find a relation between the bending moment M and the corresponding unit strain ϵ in the outer fibers of the beam, it is necessary to introduce a variable m_1, defined[3,4] as the statical moment, about the σ-axis, of an area under the stress-strain curve, taken to a variable point A on the curve, provided however that the variable base ϵ is reduced to unity (Fig. 3). The moment about the σ-axis of the area under the curve OA in Fig. 1 is evidently equal to $\int_0^\epsilon \epsilon\,\sigma\,d\epsilon$. Reducing the base ϵ to a unit length changes both the elements of area and their lever arms ϵ times; therefore, the mathematical expression defining m_1 is as follows:

$$m_1 = \frac{1}{\epsilon^2}\int_0^\epsilon \epsilon\,\sigma\,d\epsilon \qquad (1)$$

The variable m_1 may be considered as a function of ϵ. In cases when σ and ϵ are not related by a mathematical equation the value of the integral in Eq. 1 must be determined by summation, dividing the area under the stress-strain curve into a large number of sufficiently narrow strips parallel to the σ-axis.

The requisite relation between the bending moment M and the unit strain ϵ in the outer fibers of the beam is found by equating the internal moment of stresses and the external bending moment. It may be observed that for the same unit strain, ϵ, the internal moment is proportional to the breadth b and to the square of the depth h of the section, the coefficient of proportionality being simply related to m_1 in the following manner:

$$M = 2\,m_1\,b\,(\tfrac{1}{2}h)^2; \qquad \text{or} \qquad m_1 = \frac{2\,M}{b\,h^2} \qquad (2)$$

In the analysis of statically determinate rectangular beams, when the bending moment M and the dimensions of the cross section are known, Eq. 2 determines the value of the variable m_1. The unit strain ϵ can then be found from a table or graph representing the relation between m_1 and ϵ expressed by Eq. 1, and the unit stress σ in the outer fibers of the beam, by the stress-strain relation.

3. Unit Strain—Bending Moment Relation in an I-Beam or a Channel Beam

It is possible to derive a rigorous expression for a function m, applicable to I-beams and channel beam sections, which can be used in the same manner as the function m_1, defined in Section 2. This function would contain two independent parameters, a feature restricting the field of its usefulness. For this reason, it has been considered wise to widen the scope of applicability of the function m by ignoring the variation of unit stress over the areas of flanges. An analogous approximation is often made in the elastic range, which is essen-

tially the so-called flange-area method of design applied to plate girders. The error involved in this approximation in the region above the elastic limit is smaller than that involved in the elastic range. When this approximation is made, the web of the beam is assumed equal to the distance between the centers of flanges, and the areas of flanges are assumed concentrated at the extremities of the web (Fig. 4(*a*)).

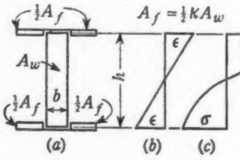


FIG. 4

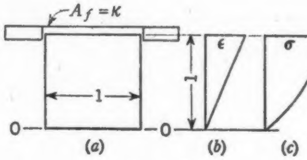


FIG. 5

To introduce the function m, imagine a section as in Fig. 5(*a*), strained and stressed as in Figs. 5(*b*) and 5(*c*). The width and depth of the web are both equal to unity, and the flange area A_f is equal to κ. The moment about the axis OO of stresses produced in this section, when it is strained as shown, is defined as the m-function. Evidently—

$$m = m_1 + \kappa\,\sigma \qquad (3)$$

In Eq. 3, m_1 is the function of ϵ defined by Eq. 1, and σ is the unit stress corresponding to the same value of ϵ, thus making m also a function of ϵ.

Combining Eqs. 1 and 3:

$$m = \frac{1}{\epsilon^2}\int_0^\epsilon \epsilon\,\sigma\,d\epsilon + \kappa\,\sigma \qquad (4)$$

The advantage of this variety of the function m over the one referred to at the beginning of this section is that it contains only one parameter κ instead of two, which widens its scope.

The function m can be easily related to the bending moment carried by the beam of Fig. 4. The web area of this beam is A_w, the depth is h, the area of one flange is A_f, and the ratio $\frac{A_f}{\frac{1}{2}A_w} = \kappa$.

Imagine a number of beams possessing the same flange-web ratio κ and strained to the same unit strain ϵ in the outside fibers. The following relations exist between the dimensions of the beams and the bending moments carried by them: When h remains constant and A_w varies, the internal moment varies in proportion to A_w. On the other hand, as long as A_w retains its value, the moment varies in proportion to h. Thus, the internal moment is proportional to the product $A_w\,h$. Therefore, the external bending moment M can be expressed, in terms of the function m, as follows:

$$M = 2\,m\left(\tfrac{1}{2}A_w\right)\left(\tfrac{1}{2}h\right) = \tfrac{1}{2}\,m\,A_w\,h \qquad (5)$$

Eq. 5 can be applied to statically determinate I-beams and channels just as Eq. 2 can be applied to rectangular beams. The function m in Eq. 5 should

be computed for the value of the parameter κ appropriate to the shape of the section.

The analysis of shear stresses requires the introduction of the function q, defined as the sum of the normal stresses developed by the section shown in Fig. 5. When the flange area κ is equal to zero, the corresponding q-function is referred to as q_1. Functions q and q_1, representing the internal normal forces, are thus analogous to the functions m and m_1, representing the internal moments, developed in the half-beam section of Fig. 5. Evidently,

$$q_1 = \frac{1}{\epsilon}\int_0^\epsilon \sigma\, d\epsilon \qquad (6a)$$

and

$$q = \frac{1}{\epsilon}\int_0^\epsilon \sigma\, d\epsilon + \kappa\, \sigma \qquad (6b)$$

4. Angle Changes

The analysis of statically indeterminate beams requires the derivation of relations between the angle changes and the unit strains and relations between the deflections and the unit strains.

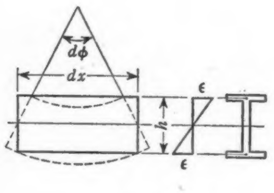


Fig. 6

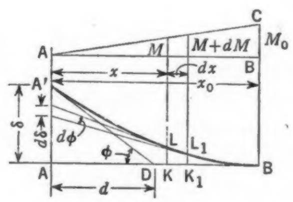


Fig. 7

Fig. 6 represents an element of length dx of a straight prismatic I-beam or channel beam with a symmetrical cross section, whose effective depth is h. It is strained to a unit strain ϵ at the centers of flanges, and the end sections rotate and form an angle $d\phi$ between them; thus:

$$d\phi = \frac{\epsilon\, dx}{\frac{1}{2} h} = \frac{2\,\epsilon\, dx}{h} \qquad (7a)$$

The angle change ϕ on a finite length of the beam is:

$$\phi = \frac{2}{h}\int \epsilon\, dx \qquad (7b)$$

to integrate Eq. 7b, ϵ and dx should be expressed in terms of the same variable.

In Fig. 7 the bending moment in the beam varies as a straight line from zero at one end to M_0 at the other end. If the shearing force in the beam is V, then, by a well-known relation:

$$dx = \frac{dM}{V} \qquad (8a)$$

Differentiating Eq. 5,

$$dM = \tfrac{1}{2} A_w h \, dm \qquad (8b)$$

Substituting Eq. 8*b* in Eq. 8*a*,

$$dx = \tfrac{1}{2} \frac{A_w h}{V} dm \qquad (8c)$$

Placing this value for dx (Eq. 8*c*) in Eq. 7*b*, the following expression for ϕ is obtained:

$$\phi = \frac{A_w}{V} \int_0^{\epsilon_0} \epsilon \, dm \qquad (9)$$

The integral in Eq. 9 represents the area between the (m-ϵ)-curve and the m-axis (Fig. 8), taken up to point A on the curve with the coordinates m_0 and

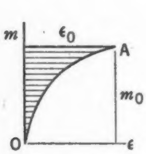


FIG. 8

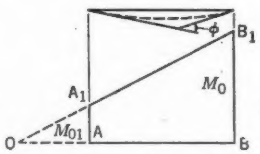


FIG. 9

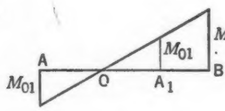


FIG. 10

ϵ_0, corresponding to the state of strain in the beam at end B with moment M_0. The integral may be considered as a function of ϵ_0. It can be computed from the (m-ϵ)-curve for different values of ϵ_0 by summing up the terms referring to a large number of narrow horizontal strips. In some special cases the integration can also be performed by calculus. The function expressed by the integral is given a symbol n, so that

$$\int_0^{\epsilon_0} \epsilon \, dm = n_0 \qquad (10)$$

Eq. 9 then becomes

$$\phi = \frac{A_w}{V} n_0 \qquad (11)$$

The loading conditions of the beam, to which Eq. 11 applies, are presented in Fig. 7. To solve for ϕ, compute m_0 from M_0 by Eq. 5. For the resulting value of m_0 select the value of the unit strain ϵ_0 from the (m-ϵ)-curve. For this value of ϵ_0 find the value of n_0 in the (n-ϵ)-curve, and then substitute it into Eq. 11. If the values of the m-functions and n-functions are plotted on the same sheet, or listed[5a] as in Table 1(*a*), the necessary n_0-value can be read directly against the corresponding m_0-value without referring first to ϵ_0.

Eq. 11 can be easily extended to a trapezoidal shape of the moment diagram by adding, algebraically, angle changes contributed by different parts of the

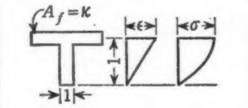
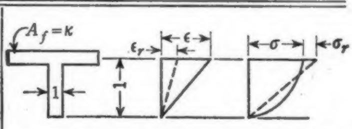
[5a] Tables 1(*a*) and 1(*b*) are greatly condensed forms of the complete tables, and they are confined to values of functions m, n, u, and $\frac{dq}{dm}$ for mild steel I-beams and channel beams with $\kappa = 1$ and $1{,}000\,\epsilon$ varying between the limits 1.1 and 65, as needed for solving the illustrative examples in the paper. The complete tables contain similar functions for mild steel computed for four values—of $\kappa = 0$, $\kappa = \frac{1}{2}$, $\kappa = 1$, and $\kappa = 1\frac{1}{2}$—and 46 values of $1{,}000\,\epsilon$ varying between the limits 1.1 and 100. The photostats of these tables are available on request at the price of $1.00.

beam. Thus, in Fig. 9, the angle change caused by the moments on the length AB is equal to the difference of the angle changes contributed by the triangular parts OBB_1 and OAA_1; and, since A_w and V are the same in both these parts,

$$\phi = \frac{A_w}{V} (n_0 - n_{01}) \qquad (12)$$

The functions n_0 and n_{01} in Eq. 12 correspond to the strains present in the outer

TABLE 1.—Strains, Stresses, and Derived Functions for Mild Steel Beams[a] ($\kappa = 1$)

1,000 ϵ	σ $\left(\frac{\text{kips}}{\text{in.}^2}\right)$	m $\left(\frac{\text{kips}}{\text{in.}^2}\right)$	 (a) Primary Values			 (b) Residual Values			
			$10^3 n$	$10^3 u$	dq/dm	$10^3 n_r$	$10^3 u_r$	1,000 ϵ_r	σ_r
1.1	33.0	44.00	24.20	709.9	1.5	0	0	0	0
1.2	33.0	44.88	25.22	754.7	1.64	0.043	1.463	0.078	−0.66
1.3	33.0	45.56	26.06	793.3	1.77	0.113	5.084	0.161	−1.17
1.4	33.0	46.10	26.79	826.8	1.91	0.222	10.09	0.247	−1.58
1.5	33.0	46.54	27.43	856.2	2.05	0.352	16.12	0.336	−1.91
1.75	33.0	47.33	28.69	915.7	2.39	0.696	32.28	0.567	−2.50
2	33.0	47.84	29.64	960.9	2.73	1.037	48.48	0.804	−2.88
3	33.0	48.76	31.86	1,067.9	4.09	2.144	101.79	1.781	−3.57
4	33.0	49.08	32.97	1,122.4	5.45	2.861	137.16	2.77	−3.81
5	33.0	49.23	33.64	1,155.1	6.82	3.343	160.82	3.77	−3.91
6	33.0	49.31	34.08	1,177.0	8.18	3,689	177.83	4.77	−3.98
7	33.0	49.36	34.40	1,192.6	9.54	3.947	190.43	5.77	−4.02
8	33.0	49.40	34.63	1,204.3	10.91	4.132	199.71	6.76	−4.05
9	33.0	49.42	34.82	1,213.5	12.27	4.292	207.6	7.76	−4.06
10	33.0	49.43	34.97	1,220.8	13.63	4.427	214.3	8.76	−4.07
11	33.0	49.44	35.09	1,226.8	15.00	4.536	219.7	9.76	−4.08
12	33.0	49.45	35.19	1,231.7	16.36	4.625	224.1	10.76	−4.09
13	33.0	49.46	35.28	1,236.0	17.72	4.697	227.7	11.76	−4.10
14	33.0	49.47	35.35	1,239.6	19.09	4.758	230.7	12.76	−4.10
15	33.0	49.47	35.41	1,242.7	20.45	4.822	233.8	13.76	−4.10
16	33.0	49.47	35.47	1,245.5	21.81	4.877	236.6	14.76	−4.10
17	33.0	49.48	35.52	1,247.9	23.18	4.914	238.4	15.76	−4.11
18	33.0	49.48	35.56	1,250	24.54	4.958	240.5	16.76	−4.11
20	34.25	50.79	60.46	2,499	1.10	28.22	1,407	18.73	−3.84
22	35.35	52.06	87.13	3,870	1.08	53.25	2,694	20.70	−3.70
24	36.45	53.33	116.34	5,409	1.07	80.79	4,145	22.67	−3.55
26	37.5	54.61	148.34	7,135	1.05	111.06	5,778	24.63	−3.46
28	38.5	55.89	182.90	9,045	1.05	143.85	7,590	26.60	−3.42
30	39.5	57.17	220.0	11,144	1.07	179.17	9,587	28.57	−3.38
32	40.4	58.33	256.0	13,222	1.09	213.4	11,568	30.54	−3.35
34	41.2	59.43	292.3	15,361	1.10	248.1	13,612	32.51	−3.37
36	42.0	60.52	330.4	17,650	1.12	284.7	15,803	34.49	−3.39
38	42.7	61.47	365.6	19,794	1.16	318.4	17,858	36.46	−3.40
40	43.3	62.35	399.9	21,920	1.16	351.3	19,896	38.44	−3.46
45	45.0	64.68	498.9	28,200	1.09	446.6	25,950	43.38	−3.51
50	46.8	67.11	614.4	35,810	1.10	558.1	33,290	48.32	−3.53
55	48.3	69.21	724.6	43,330	1.19	664.7	40,570	53.27	−3.61
60	49.6	71.06	831.0	50,790	1.24	767.8	47,800	58.22	−3.70
65	50.75	72.75	936.6	58,390	1.20	870.4	55,180	63.18	−3.81

[a] Units are as follows: m, $10^3 n$, $10^3 n_r$, σ, and σ_r are in kips (thousand pounds) per square inch; and $10^3 u$ and $10^3 u_r$ are in $\frac{\text{kips}^2}{\text{in}^4}$.

fibers at points B and A, respectively. Eq. 12 is also applicable to the setup of Fig. 10, in which the bending moment changes its sign on the length AB, because the resultant angle change over the length AA_1 is zero.

Eq. 11 can be given a different form by substituting $\frac{M_0}{x_0}$ for the shear V and using Eq. 5:

$$V = \frac{M_0}{x_0} = \frac{A_w\, h\, m_0}{2\, x_0} \quad \text{(13a)}$$

Substituting Eq. 13*a* into Eq. 11:

$$\phi = \frac{2}{h}\, x_0 \left(\frac{n_0}{m_0}\right) \quad \text{(13b)}$$

The ratio $\frac{n_0}{m_0}$ can be calculated as a new function. Eq. 13*b* is less convenient for computing the angle changes than is Eq. 11, but it will be used to advantage subsequently in reaching certain theoretical conclusions.

5. Deflections

Referring again to Fig. 7, end B, at which the moment is M_0, is assumed to be fixed; and it is necessary to find an expression for the deflection of the other end, A, where the moment is zero. Consider an element dx of this beam, a distance x from the left end. The angular change $d\phi$ occurring on the length of this element causes a deflection $d\delta$ of the left end, so that

$$d\delta = x\, d\phi \quad \text{(14a)}$$

The total deflection of the left end is then

$$\delta = \int_0^{x_0} x\, d\phi \quad \text{(14b)}$$

which is the same expression as in the elastic theory. By Eq. 7*a*, $d\phi = \frac{2\,\epsilon\, dx}{h}$. Consequently,

$$\delta = \frac{2}{h} \int_0^{x_0} x\, \epsilon\, dx \quad \text{(15)}$$

From Eq. 5 and Fig. 7,

$$x = \frac{M}{V} = \tfrac{1}{2}\, A_w\, h\, \frac{m}{V} \quad \text{(16)}$$

Substituting Eqs. 8*c* and 16 in Eq. 15:

$$\delta = \frac{h}{2} \left(\frac{A_w}{V}\right)^2 \int_0^{\epsilon_0} m\, \epsilon\, dm = \frac{h}{2} \left(\frac{A_w}{V}\right)^2 u_0 \quad \text{(17a)}$$

in which the integral—

$$u_0 = \int_0^{\epsilon_0} m\, \epsilon\, dm \quad \text{(17b)}$$

—represents a new function which signifies the statical moment about the ϵ-axis of the area between the (m-ϵ)-curve and the m-axis taken to point A on the curve with coordinates ϵ_0 and m_0 (Fig. 8). The function u_0 can be evaluated

by summation. The function u_0 in Eq. 17b pertains to the strain condition existing at point B in the beam, where the bending moment is M_0.

The other version of this equation analogous to Eq. 13b is:

$$\delta = \frac{2\,x^2{}_0}{h}\,\frac{u_0}{m^2{}_0} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (18)$$

In the following discussion and tables the subscript 0 in the symbols of n_0 and u_0 will be omitted, as a rule, when no specific point of the beam is under consideration.

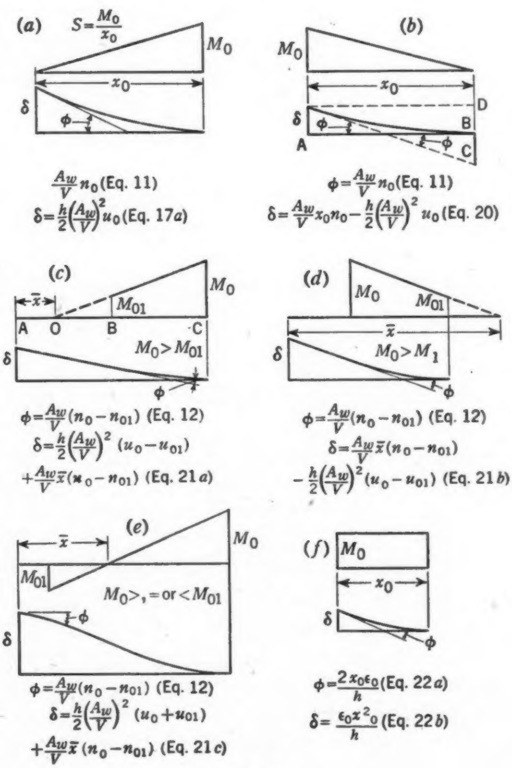


FIG. 11

Formulas for the deflections, corresponding to various types of trapezoidal moment diagrams, are summarized in Fig. 11. They can be easily developed either by the method used to derive Eq. 17a, or from simple geometrical considerations, on the basis of Eqs. 11 and 17a which are fundamental. This condition is illustrated by Fig. 11(b). Let it be required to find the deflection at point A of a beam with a moment M_0, in relation to the tangent at point B, where the moment is zero. The deflection, of course, is assumed to be small. By Eq. 11:

$$\overline{\mathrm{CD}} = x_0\,\phi = \frac{A_w}{V}\,x_0\,n_0 \ldots\ldots\ldots\ldots\ldots\ldots \quad (19a)$$

By Eq. 17*a*

$$\overline{BC} = \frac{h}{2}\left(\frac{A_w}{V}\right)^2 u_0 \qquad (19b)$$

Finally:

$$\delta = \overline{CD} - \overline{BC} = \frac{A_w}{V} x_0\, n_0 - \frac{h}{2}\left(\frac{A_w}{V}\right)^2 u_0 \qquad (20)$$

It is necessary to remember that the two cases dealing with the triangular moment diagrams (Figs. 11(*a*) and 11(*b*)) are quite distinct. Therefore, the corresponding formulas for deflections, Eqs. 17*a* and 20, cannot be converted from one to the other by direct algebra; and the same is true regarding the three trapezoidal cases, Figs. 11(*c*), 11(*d*), and 11(*e*). The deflections for these cases are, respectively:

For case (*c*)—

$$\delta = \frac{h}{2}\left(\frac{A_w}{V}\right)^2 (u_0 - u_{01}) + \frac{A_w \bar{x}}{V} (n_0 - n_{01}) \qquad (21a)$$

for case (*d*)—

$$\delta = \frac{A_w \bar{x}}{V} (n_0 - n_{01}) - \frac{h}{2}\left(\frac{A_w}{V}\right)^2 (u_0 - u_{01}) \qquad (21b)$$

and, for case (*e*)—

$$\delta = \frac{h}{2}\left(\frac{A_w}{V}\right)^2 (u_0 + u_{01}) + \frac{A_w \bar{x}}{V} (n_0 - n_{01}) \qquad (21c)$$

In the first two trapezoidal cases the sign of the moment does not change, but in the last case it does change. The triangular cases can be obtained from the corresponding trapezoidal cases by assuming one of the end moments to be zero.

In Eqs. 11, 12, 17*a*, 20, and 21, derived for Figs. 11(*a*) to 11(*e*), the quantities A_w, V, n, and u must always be considered as positive; the same condition is true for x_0 and $\bar{x}$, except that $\bar{x}$ in Fig. 11(*c*) may either represent a positive quantity (when point A lies outside the region OB) or a negative quantity (when point A is within the distance OB). Moments M are considered positive when there is tension in the bottom fibers, and the angle changes caused by positive moments are taken as positive. Deflections δ are assumed positive, if they are above the tangent of reference. The quantities n_0 and u_0 refer to the points with the moments M_0, and the quantities n_{01} and u_{01} to the points with the moments M_{01}.

Curvilinear moment diagrams can be treated by dividing the curves into a number of sufficiently narrow trapezoids and applying appropriate equations. When a large number of trapezoids is necessary, the computation is quite laborious.

A special case, which cannot be solved by the foregoing equations, Eqs. 11, 12, 17*a*, 20, 21, and 22, involves the rectangular moment curve shown as Fig. 11(*f*). In this case the flange strain ϵ_0 is constant all along the beam and, by Eq. 7*b*:

$$\phi = \frac{2\, x_0\, \epsilon_0}{h} \qquad (22a)$$

and

$$\delta = \tfrac{1}{2}\, x_0\, \phi = \frac{\epsilon_0\, x^2{}_0}{h} \quad \text{(22}b\text{)}$$

Angle changes occurring in the elastic range are proportional to bending moments, which follows from the well-known formula:

$$d\phi = \frac{M}{E\, I}\, dx \quad \text{(23}a\text{)}$$

and the moment diagram may thus be considered as a scale picture of the angle changes. At the same time:

$$d\phi = \frac{2\, \epsilon}{h}\, dx \quad \text{(23}b\text{)}$$

These relations will be utilized in Section 8, in computations pertaining to the elastic range.

Beyond the elastic range the angle changes increase much faster than the moments, and the moment diagram does not represent the angle changes to scale. Referring to Fig. 7, point D, where the tangents at the ends of the deflected beam meet, is a distance $\bar{d}$ from the point of zero moment, point A. By geometry and by Eqs. 13*b* and 18:

$$\bar{d} = \frac{\delta}{\phi} = x_0\, \frac{u_0}{m_0\, n_0} \quad \text{(24)}$$

In the elastic range, $\bar{d} = \tfrac{2}{3}\, x_0$, but beyond it $\bar{d}$ increases and may even approach x_0.

Once the expressions for the deflections and angle changes in beams are known, the indeterminate stress analysis outside the elastic range is in no way different from conventional analysis, and it merely consists in setting up the equations between the deflections and the angle changes prescribed by the conditions of restraint.

6. Shearing Stresses in Beams

The greatest shearing stress existing on a normal or a longitudinal plane of an I-beam is present in the web of the beam at the neutral axis, and an expression will be derived for the value of this stress τ_0, following the method used in the elastic analysis.

Fig. 12 represents one half of the cross section of an I-beam and the free body diagram of the length dx of this section. The greatest strains at the left and right sides of this section are ϵ and $(\epsilon + d\epsilon)$, respectively. Recalling the definition of the function q in Eq. 6*b*, the following expression is obtained from the condition of equilibrium of horizontal forces:

$$\tau_0\, b\, dx = \tfrac{1}{2}\, A_w\, [q\,(\epsilon + d\epsilon) - q(\epsilon)] = \tfrac{1}{2}\, A_w\, dq \quad \text{(25}a\text{)}$$

and

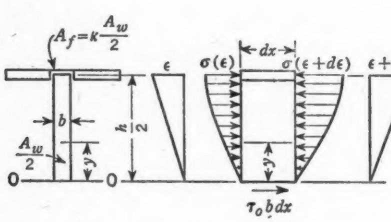
$$\tau_0 = \frac{h}{2}\, \frac{dq}{dx} \quad \text{(25}b\text{)}$$

If Eq. 8*c* is substituted for dx in Eq. 25*b*:

$$\tau_0 = \frac{V}{A_w}\frac{dq}{dm} \qquad \text{(25c)}$$

The function $\frac{dq}{dm}$ must be computed point by point, by taking increments of the functions q and m corresponding to the same increments in ϵ.

Thus, shearing stresses in inelastic range are dependent not only on the shearing force V, but also on the bending moment, since the latter determines the strain ϵ, and the derivative $\frac{dq}{dm}$ is a function of ϵ. The ratio $\frac{V}{A_w}$ represents uniform shearing stress per square unit of area of the web. The function $\frac{dq}{dm}$ therefore signifies a coefficient allowing for nonuniformity of shear distribution over the web.



(*a*) (*b*)

Fig. 12

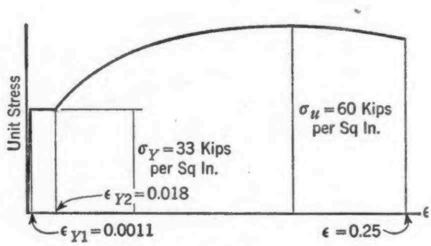


Fig. 13

A similar procedure leads to a more general formula for the value of the horizontal or vertical shearing stress at a distance y from the neutral axis (Fig. 12):

$$\tau = \frac{V}{A_w}\,\frac{d\left[q(\epsilon) - \frac{y}{\frac{1}{2}h}\,q_1\left(\frac{y}{\frac{1}{2}h}\,\epsilon\right)\right]}{dm(\epsilon)} \qquad \text{(25d)}$$

In Eqs. 25*a* and 25*d*, $q(\epsilon)$ and $m(\epsilon)$ signify the values of the functions q and m corresponding to the value ϵ of the strain in the outer fibers of the beam, whereas $q_1\left(\frac{y}{\frac{1}{2}h}\,\epsilon\right)$ represents the value of the function q_1 corresponding to the value of the strain $\frac{y}{\frac{1}{2}h}\,\epsilon$.

7. Functions m, n, u, and $\frac{dq}{dm}$ for Mild Steel Beams

The foregoing bending theory will be applied to mild steel beams, whose stress-strain curve is presented in Fig. 13 and in Table 1(*a*). The main mechanical properties of this steel satisfy the material requirements of the 1940 Specifications of the American Railway Engineering Association for fixed-span railway bridges. They are as follows:

Unit Stresses, in Kips Per Square Inch—	
Yield point	33
Ultimate strength	60
Modulus of elasticity	30,000
Unit strain (%) at the beginning of yielding	0.11
Percentage elongation in 8 in.	25

In comparison with the actual curve, the stress-strain curve in Fig. 13 is idealized in two respects: First, the elastic section of the curve is assumed to continue straight to the yield point without any curved transition; and, second, the yield section is assumed to extend horizontally to the point where the strain hardening begins—at $\epsilon = 1.8\%$. These approximations do not distort the mechanical behavior of beams materially, and they are introduced merely for the purpose of reducing the labor of computation and not because they are inherent in the theory presented.

The necessary derived curves are computed for four values[5a] of the parameter $\kappa - 0, \frac{1}{2}, 1$, and $1\frac{1}{2}$. The first of these values corresponds to a rectangular beam, whereas the remaining three cover the range of standard I-beams. Formulas pertaining to the elastic and plastic parts of the derived curves (that is, the parts within the limits of ϵ from zero to 0.11% and from 0.11% to 1.8%, respectively) can be developed by algebra as demonstrated in Sections 8 and 9.

8. Elastic Part

In the elastic range, $\sigma = E\,\epsilon$. Substituting this value in Eq. 4 and integrating:

$$m = (\kappa + \tfrac{1}{3})\,\sigma = E\,(\kappa + \tfrac{1}{3})\,\epsilon \qquad (26a)$$

Differentiating Eq. 26*a* and substituting *m* and *dm* in Eqs. 10 and 17*b* for *n* and *u*:

$$n = \tfrac{1}{2}\,E\,(\kappa + \tfrac{1}{3})\,\epsilon^2 = \tfrac{1}{2}\,(\kappa + \tfrac{1}{3})\,\sigma\,\epsilon \qquad (26b)$$

and

$$u = \tfrac{1}{3}\,E^2\,(\kappa + \tfrac{1}{3})^2\,\epsilon^3 = \tfrac{1}{3}\,(\kappa + \tfrac{1}{3})^2\,\sigma^2\,\epsilon \qquad (26c)$$

Similarly, by Eq. 6*b*,

$$q = (\kappa + \tfrac{1}{2})\,E\,\epsilon \qquad (27a)$$

and

$$\frac{dq}{dm} = \frac{\kappa + \frac{1}{2}}{\kappa + \frac{1}{3}} \qquad (27b)$$

Eqs. 26 and 27 are valid to the beginning of yielding, where the coordinates of the stress-strain curve are $\epsilon = \epsilon_{Y1}$ and $\sigma = \sigma_y$.

The specific values of these functions at the yield point under the conditions of the assumed stress-strain curve are as follows:

$$m = 33\,(\kappa + \tfrac{1}{3})\left(\frac{\text{kips}}{\text{in.}^2}\right) \qquad (28a)$$

$$n = 0.01815\,(\kappa + \tfrac{1}{3})\left(\frac{\text{kips}}{\text{in.}^2}\right) \qquad (28b)$$

and

$$u = 0.3993\ (\kappa + \tfrac{1}{3})^2 \left(\frac{\text{kips}^2}{\text{in.}^4}\right) \qquad (28c)$$

Expressions for the angle change and deflection in the elastic range, obtained by substituting Eqs. 26*b* and 26*c* in Eqs. 11 and 17*a*, and simplifying, are:

$$\phi = \frac{x\,\epsilon}{h} \qquad (29a)$$

and

$$\delta = \tfrac{2}{3}\,\frac{x^2\,\epsilon}{h} \qquad (29b)$$

9. Plastic Part

In the plastic range, $\sigma = \sigma_y$ = constant, as ϵ varies between the limits of ϵ_{Y1} and ϵ_{Y2}. Although the expression for the *m*-function can be obtained from the basic formula, Eq. 4, it is derived more conveniently from the first principles (Fig. 14) by taking moments about the axis OO of the normal stresses distributed over the section as shown:

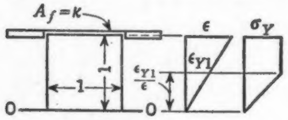


Fig. 14

$$m = \kappa\,\sigma_Y(1) + \sigma_Y(1)\tfrac{1}{2} - \frac{\sigma_Y}{2}\,\frac{\epsilon_{Y1}}{\epsilon}\,\frac{1}{3}\,\frac{\epsilon_{Y1}}{\epsilon} = (\kappa + \tfrac{1}{2})\,\sigma_Y - \frac{\sigma_Y}{6}\left(\frac{\epsilon_{Y1}}{\epsilon}\right)^2 \qquad (30a)$$

By Eq. 10,

$$n = \int_0^{\epsilon_{Y1}} \epsilon\, dm + \int_{\epsilon_{Y1}}^{\epsilon} \epsilon\, dm \qquad (30b)$$

By Eq. 26*b*,

$$\int_0^{\epsilon_{Y1}} \epsilon\, dm = \frac{E}{2}\,(\kappa + \tfrac{1}{3})\,\epsilon^2_{Y1} \qquad (30c)$$

From Eq. 30*a*,

$$dm = \frac{\sigma_Y}{3}\,\frac{\epsilon^2_{Y1}}{\epsilon^3}\,d\epsilon \qquad (31)$$

Eq. 31 makes it possible to evaluate the second integral in the expression for *n*. Then:

$$n = E\,\epsilon^2_{Y1}\left[\tfrac{1}{2}\,(1 + \kappa) - \tfrac{1}{3}\,\frac{\epsilon_{Y1}}{\epsilon}\right] \qquad (32a)$$

By applying the same procedure to *u*,

$$u = \tfrac{1}{3}\,E^2\,\epsilon^3_{Y1}\left[\left(\kappa^2 + \frac{5}{3}\,\kappa + \frac{5}{9}\right) - (\kappa + 0.5)\,\frac{\epsilon_{Y1}}{\epsilon} + \frac{1}{18}\left(\frac{\epsilon_{Y1}}{\epsilon}\right)^3\right] \qquad (32b)$$

From Fig. 14,

$$q = (\kappa + 1)\,\sigma_Y - \frac{\sigma_Y}{2}\,\frac{\epsilon_{Y1}}{\epsilon} \qquad (32c)$$

Then, by differentiating Eqs. 30*a* and 32*c* and simplifying:

$$\frac{dq}{dm} = \frac{3}{2} \frac{\epsilon}{\epsilon_{Y1}} \quad \text{(33)}$$

At the beginning of yielding, where $\epsilon = \epsilon_{Y1}$, $\frac{dq}{dm} = \frac{3}{2}$, as in a rectangular beam. At the end of yielding, where $\epsilon = \epsilon_{Y2} = \frac{1.8}{0.11}$, $\frac{dq}{dm} = 24.54$.

10. Strain-Hardening Part

Values of the derived functions above the plastic range can be computed only by successive summation of the increments of integrals, with all the neces-

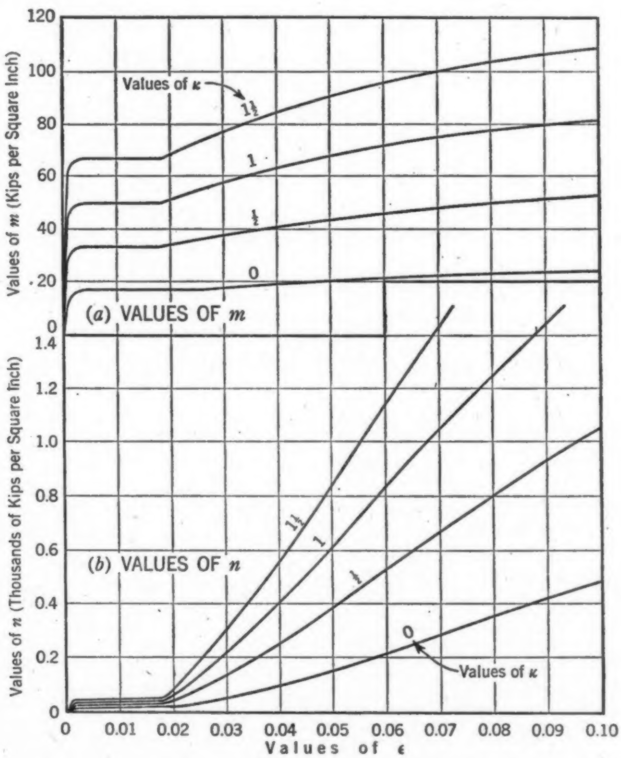


Fig. 15.—Constants for the Design of Mild Steel Beams

sary values properly tabulated in a systematic manner. The method is illustrated by an example of quantities referring to $\kappa = 1$ and $\epsilon = 0.020$. First, the necessary values of functions corresponding to the value of $\epsilon = 0.018$, immediately below $\epsilon = 0.020$, are found either by integration, or by step-by-step summation.

For $\epsilon = 0.018$.—

Solving Eqs. 30*a* and 32: $m = 49.48$ kips per sq in.; $n = 0.03556$ kips per sq in.; $u = 1.250 \dfrac{\text{kips}^2}{\text{in.}^4}$; and $q = 64.99$ kips per sq in. Furthermore, $\displaystyle\int_0^{0.018} \sigma\, d\epsilon = 0.576$ kips per sq in.; and $\displaystyle\int_0^{0.018} \epsilon\, \sigma\, d\epsilon = 0.005340$ kips per sq in.

For $\epsilon = 0.020$.—

The stress-strain curve gives $\sigma = 34.25$ kips per sq in. for $\epsilon = 0.020$, from which $\displaystyle\int_0^{0.020} \epsilon\, \sigma\, d\epsilon = 0.005340 + \tfrac{1}{2}(0.018 + 0.020) \times \tfrac{1}{2}(33 + 34.25) \times 0.002 = 0.006618$ kips per sq in. By Eq. 4, $m = \dfrac{0.006618}{0.020^2} + 1 \times 34.25 = 50.79$ kips per sq in. By Eq. 10, adding an increment to the value of n at $\epsilon = 0.018$, $n = 0.03556 + \tfrac{1}{2}(0.018 + 0.020) \times (50.79 - 49.48) = 0.006046$ kips per sq in. Similarly, by Eq. 17*b*, $u = 1.250 + \tfrac{1}{2}(49.48 + 50.79) \times \tfrac{1}{2}(0.018 + 0.020) \times (50.79 - 49.48) = 2.499 \dfrac{\text{kips}^2}{\text{in.}^4}$. As before, $\displaystyle\int_0^{0.020} \sigma\, d\epsilon = 0.576 + \tfrac{1}{2}(33 + 34.25) \times 0.002 = 0.6432$ kips per sq in.; and, by Eq. 6*b*, $q = \dfrac{0.6432}{0.020} + 1 \times 34.25 = 66.41$ kips per sq in. Finally, $\dfrac{dq}{dm} = \dfrac{66.41 - 64.99}{50.79 - 49.48} = 1.10$. This value of $\dfrac{dq}{dm}$ should be attributed to the value of independent variable $\epsilon = 0.019$.

The functions m, n, and u are computed to the value of $\epsilon = 0.100$, for the four values of the parameter κ, and the results of computation are presented in

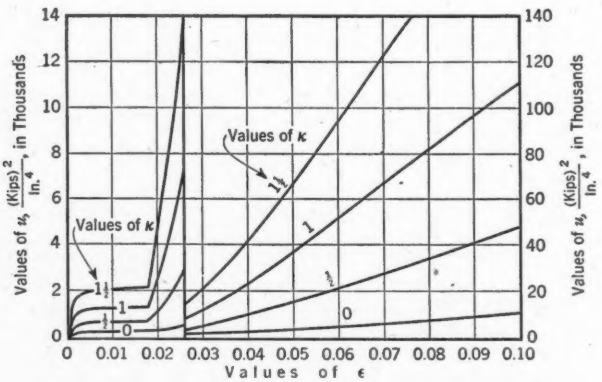


FIG. 16.—CURVES FOR THE CONSTANT u, IN MILD STEEL BEAMS

Figs. 15 and 16 and, in a greatly expanded form, in Table 1(*a*). The graphs, not being very accurate, serve merely as illustrations, whereas the more comprehensive table, of which Table 1(*a*) is a part, is intended for use in the analysis.[5a]

11. Effect of Shear Stresses

Only one set of values of the shear coefficient $\frac{dq}{dm}$, corresponding to $\kappa = 1$, is given in Fig. 17 and in Table 1(*a*), in view of the comparative unimportance of this coefficient in design. Some comments are needed on the numerical values of this coefficient, however. Everywhere outside the plastic range (that is, outside the limits of ϵ from 0.0011 to 0.018) this coefficient has a magnitude of from approximately 1.1 to 1.3, which signifies that the greatest shearing stress in the web of the beam is only from about 10% to 30% higher than the average stress per unit area of the web. In the plastic range, however, this coefficient varies independently of the value of κ, and in a straight line from the value 1.5 at $\epsilon = \epsilon_{Y1}$ to a very high value of 24.54 at $\epsilon = \epsilon_{Y2}$.

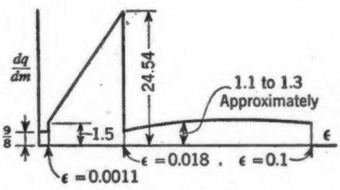


Fig. 17 (Not to Scale)

The correctness of this rather startling phenomenon is made plain by reference to Fig. 18 representing the free body diagram of a small element of an I-beam. Since normal stresses on the parts of the sections BC and B_1C_1 are constant and equal, no shearing stresses acting either on horizontal or on vertical planes are possible in these outer parts of the web BC, all shear on the cross section thus being localized on a small central part of the web BB′. The intensity of stress on BB′, indicated by a dotted parabola, may rise very high, because the resultant shearing stress over the cross section must equal the shearing force.

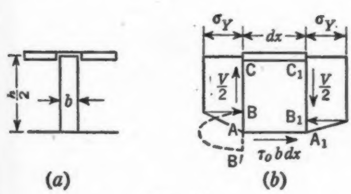


Fig. 18

Of course, this deduction is only as authentic as the underlying assumption of plane cross sections. If the cross sections warp (which seems inevitable in the presence of high shearing stresses), some redistribution of stresses and strains, both normal and tangential, must follow—a condition which raises some doubts as to the validity of the present theory of bending in the plastic range.

In appraising the effect of this stress irregularity, several aspects of the situation will be distinguished and discussed: (1) The high value of shear stresses (2) the redistribution of normal stresses; and (3) the additional angle changes caused by warping.

(*1*) *High Value of the Shearing Stress.*—Shear stresses reach an intensity of 24.5 times the average stress per unit area of the web. From general considerations it appears certain that, if the absolute value of this stress is high, it is relieved by warping, perhaps to the level of yielding, because a high concentration of stress, sharply localized over a small area, is inconceivable in the absence of concentrated loads.

(*2*) *Redistribution of Normal Stresses in the Planes of Cross Sections Produced by Warping.*—Although this effect is undoubtedly present, it is likely to be small since, in the plastic and strain-hardening regions, only a slight change in the normal stresses is possible, even after an appreciable change in the strains

(*3*) *Additional Angle Changes Caused by Warping.*—Such angle changes influence the values of static unknowns and, through them, affect the general state of stress and deformation over the entire structure. The significance of this factor may be appraised indirectly by estimating the length of the beam, over which the stress irregularity considered may possibly occur. In this connection, it must be realized that, if the average unit shearing stress in the web is high, the shearing force is great, and the bending moment builds up quickly along the span, so that the plastic range of normal stresses is passed over in a short length of beam. On the other hand, if the plastic state occupies a great length of the beam, the shearing force (and with it the average shearing stress in the web) must be small, and even a high coefficient of nonuniformity would not raise the maximum shearing stress beyond the yield point.

By using appropriate values for Δq and τ_0 in Eq. 25*b* the greatest length, on which the warping is possible in the most extreme case is of the order of $\Delta x = \frac{1}{4} h$. This length appears too small to produce an appreciable effect on the static unknowns.

The foregoing material seems to warrant the conclusion that the stress irregularity occurring in the plastic region of normal stresses is too small to affect the strength of beams noticeably or to interfere seriously with the flexural theory based on linearity of the cross sections. Therefore, in Sections 14, 15, and 16 no allowance will be made for the warping of cross sections in the plastic region.

12. Stress Recession

In many important construction materials the stress-strain curve of unloading, in straight tension or compression, may be closely approximated by a straight line YO_1Y_1 parallel to the elastic part of the curve OA (Fig. 19). The

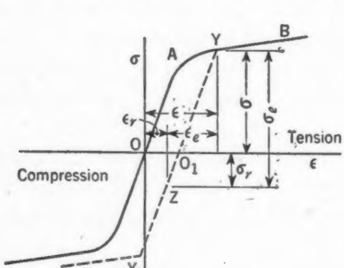


Fig. 19

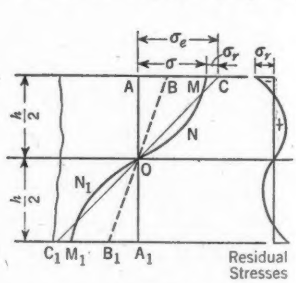


Fig. 20

new elastic straight line is preserved on reloading; and at the limit of the first loading, indicated by the point Y, it bends and merges into the continuation of the original curve YB. A residual deformation equal to $\overline{OO_1}$ remains after unloading. The proportional limit is raised to points Y and Y_1, higher than

before, whereas the modulus of elasticity retains its former value. These experimental facts, together with the assumptions stated in Section 1, form the basis of the theory of bending past the elastic limit when there is a stress recession.

Imagine a beam subjected to bending, whose normal stresses in the section AA_1 Fig. 20 are represented by the curve $MNON_1M_1$. If there is a decrease in the bending moment, the assumption of plane cross sections, combined with the mechanics of stress recession, explained in Fig. 19, demands that the decrease in stresses should be linear, so that the remaining stresses correspond to the difference of ordinates between the original curve $MNON_1M_1$ and an appropriate straight line BB_1. Further reduction in moment causes the straight line BB_1 to move farther from AA_1; and, when the moment is brought to zero, the residual stresses are represented by the difference between the ordinates of the curves $MNON_1M_1$ and CC_1, whose statical effect is zero.

An important deduction is that removal of bending moment from the beam does not result in the disappearance of all stress or of all strain. If point Y in Fig. 19 represents the state of stress in the outer fibers of the beam when the loads are first applied, the removal of the moment causes a reduction in stress σ by an amount σ_e, which brings the state of stress to point Z, Fig. 19, with a residual stress σ_r and strain ϵ_r. The presence of the latter is accompanied by a residual angle change:

$$d\phi_r = 2\frac{\epsilon_r}{h}dx \qquad (34)$$

Let the bending moment in the beam created by applied loads be M, then the corresponding m-function is computed by Eq. 5. The elastic strain ϵ_e and stress σ_e superimposed on the outer fibers of the beam by a complete removal of the moment, are found by Eq. 26a, using this value of m:

$$\epsilon_e = \frac{m}{E(\kappa + \frac{1}{3})} \qquad (35a)$$

and

$$\sigma_e = E\,\epsilon_e = \frac{m}{\kappa + \frac{1}{3}} \qquad (35b)$$

The residual strain and stress in the outside fibers of the beam are:

$$\epsilon_r = \epsilon - \frac{m}{E(\kappa + \frac{1}{3})} \qquad (36a)$$

and

$$\sigma_r = \sigma - \frac{m}{\kappa + \frac{1}{3}} \qquad (36b)$$

The residual strain ϵ_r has the same sign as the original strain ϵ, whereas the residual stress σ_r proves to be negative (that is, of the opposite sign from σ). These facts follow clearly from Figs. 19 and 20.

The angle change and the deflection caused by the release of bending moment, in the case of the simple triangular moment diagram in Fig. 7, are found by substituting the expression for strain caused by the moment removal, Eq.

35*a*, in the equations for the elastic angle change and deflection, Eqs. 29*a* and 29*b*:

$$\phi_e = \frac{x\,m}{E\,h\,(\kappa + \frac{1}{3})} \quad \text{.....................} (37a)$$

and

$$\delta_e = \tfrac{2}{3}\frac{x^2\,m}{E\,h\,(\kappa + \frac{1}{3})} \quad \text{.....................} (37b)$$

The form of Eqs. 37 is not entirely convenient. Replacing x by $\frac{M}{V}$ and substituting for the moment its expression in terms of the m-function, Eq. 5, the following expressions result:

$$\phi_e = \tfrac{1}{2}\frac{A_w}{V}\frac{m^2}{E(\kappa + \frac{1}{3})} \quad \text{...................} (38a)$$

and

$$\delta_e = \frac{h}{6}\left(\frac{A_w}{V}\right)^2\frac{m^3}{E(\kappa + \frac{1}{3})} \quad \text{..................} (38b)$$

Subtracting Eqs. 38 from the corresponding deformations brought about by creation of the moments, Eqs. 11 and 17*a*, the residual values, ϕ_r and δ_r, remaining after complete disappearance of moments, are found to be:

$$\phi_r = \frac{A_w}{V}\left[n - \tfrac{1}{2}\frac{m^2}{E(\kappa + \frac{1}{3})}\right] \quad \text{...............} (39a)$$

and

$$\delta_r = \frac{h}{2}\left(\frac{A_w}{V}\right)^2\left[u - \tfrac{1}{3}\frac{m^3}{E(\kappa + \frac{1}{3})}\right] \quad \text{.............} (39b)$$

The values of the functions m, n, and u in Eqs. 39 refer to the state of stress at the end of beam B, Fig. 7, under full loading. Expressions in the square brackets do not depend on the dimensions of the beam, except for the parameter κ, and they may be considered as the new derived functions n_r and u_r, defined as follows:

$$n_r = n - \tfrac{1}{2}\frac{m^2}{E(\kappa + \frac{1}{3})} \quad \text{...................} (40a)$$

and

$$u_r = u - \tfrac{1}{3}\frac{m^3}{E(\kappa + \frac{1}{3})} \quad \text{...................} (40b)$$

Eqs. 40 can be used for the determination of residual deflections and angle changes in exactly the same fashion as the original functions n and u are used for computing the deformations produced by creation of the moments. For this reason, Eqs. 11, 12, 17*a*, 18, 20, and 21 hold equally true for the residual deformations in their respective conditions, if the functions n and u are replaced by the corresponding functions n_r and u_r. In the elastic range the residual functions are evidently zero. Numerical values of the functions ϵ_r, σ_r, n_r, and u_r computed for different values[5a] of the independent variable ϵ and the value of the parameter $\kappa = 1$ are presented in Table 1(*b*).

When a residual deformation occurs in a statically determinate structure, it meets with no impediment and materializes freely. The situation is different

in statically indeterminate structures, in which the conditions of restraint make a free adjustment of the beam to a "no moment" state impossible. As a result, residual moments are created when the structure is unloaded, and the stresses and strains caused by the residual moments, are added to "no moment" stresses and strains. Thus, a "no loading" condition following the removal of load in a statically indeterminate beam must not be confused with the "no moment" condition. The transition from a fully loaded condition to "no loading" condition is probably always elastic, and can be analyzed by the usual formulas of the elastic theory. Since this last statement is quite important for further discussion, particularly in the analysis of moving loads, it is worthy of some elaboration. It can be expressed in the following shorthand form:

$$(\text{ALS}) - (\text{ELS}) = (\text{RS}) \quad \ldots\ldots\ldots\ldots\ldots\ldots (41)$$

in which (ALS) signifies the actual state of stress in the loaded structure, including moments, stresses, and strains, found by the theory presented herein; (ELS) denotes the state of stress under the same loading, found by the usual formulas of the strength of materials (although it is known that the moments so found project beyond the elastic range); and (RS) represents the residual state of stress after removal of all load. The minus sign in the left-hand side of Eq. 41 signifies the ordinary algebraic subtraction of the moments, stresses, and strains pertaining to the two states.

Eq. 41 holds true also when the beam has been pre-stressed past the elastic limit by one or several applications of loads, including movable loads, previous to the application of the load whose effect is being considered. In this case, (ALS) should signify the state created as a result of all loadings, including the last one; (RS) represents what remains after the last load has been removed; and (ELS) indicates the elastic stress condition created by the last loading alone. The details of application of this principle to movable loads will be explained in Example 3, Section 16.

13. Computation of Shape Constants of an I-Beam

The computation is illustrated by an example involving the standard I-beam (18 I 54.7) shown in Fig. 21: $h = 18.00 - \frac{1}{2}(0.46 + 0.92) = 17.31$ in.;

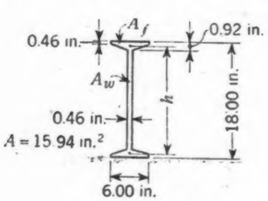


Fig. 21

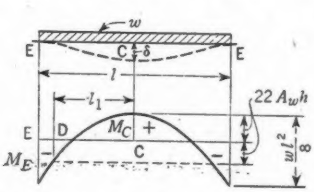


Fig. 22

$A_w = 0.46\,(17.31) = 7.96$ in.^{2}; $A_f = \frac{1}{2}(15.94 - 7.96) = 3.99$ in.^{2}; and $\kappa = \dfrac{3.99}{\frac{1}{2}(7.96)} = 1.00$.

The following examples are solved in general terms using a mild steel I-beam with $\kappa = 1$. If specific values are needed, pertaining to an 18 I 54.7, they can

be obtained by substituting the foregoing values of h and A_w in the formulas derived.

14. Example 1

Given a uniformly-loaded, fixed-ended, I-beam with $\kappa = 1$ (Fig. 22), find the value of the load (referred to as the capacity load) for which the stress at the center just reaches the yield point. For this value of the load find the stress conditions at the ends and the deflection at the center. Find also the residual state (RS) brought about by removal of this load.

Because of symmetry, there is only one static unknown, the moment at the end M_E. This is larger than the moment at the center, M_C, and therefore, at some point near the end, such as point D, the moment is equal to the moment at the center.

The method of solution consists in assuming an appropriate value for M_E, or better, in assuming an appropriate value for the maximum strain at the end ϵ_E, and on this assumption determining the angle change produced on one half of the length of the beam. This angle change should be equal to zero.

From Table 1(a) and Eq. 5, the following conditions exist at points C and D: $1{,}000\,\epsilon = 1.1$; $\sigma = 33$ (kips/in.²); $m = 44$ (kips/in.²); $1{,}000\,n = 24.2$ (kips/in.²); and $M = 22$ (kips/in.²) $A_w\,h$. The units accompanying the expression for M actually belong to the numerical coefficient; and, in combination with the units of A_w and h, they produce the proper moment units.

Trial 1.—At point E, Fig. 22, assume $1{,}000\,\epsilon_E = 22$. Then, by Table 1(a), $m_E = 52.06$ (kips/in.²); $1{,}000\,n_E = 87.13$ (kips/in.²); and $M_E = 26.03$ (kips/in.²) $A_w\,h$.

By the properties of a parabola, the length $l_1 = \overline{DC}$, over which the stress is elastic, is found as follows: $l_1 = \frac{1}{2}\,l\sqrt{\dfrac{22+22}{22+26.03}} = 0.4787\,l$; then, $\overline{ED} = x = 0.0213\,l$.

Assuming that the moment curve on the length ED is straight, the shearing force on this length is $V = \dfrac{(26.03 - 22)\,A_w\,h}{0.0213\,l} = 189.0\ (\text{kips/in.}^2)\ \dfrac{A_w\,h}{l}$.

Remembering that in the elastic region $\dfrac{M}{E\,I} = \dfrac{2\,\epsilon}{h}$, the angle change occurring on the elastic part DC is $\phi_{DC} = \dfrac{2}{h}\,(10^{-3})\,[\frac{2}{3}\,(2)(1.1)(0.4787)\,l - (1.1)(0.4787)\,l] = 0.351\,(10^{-3})\,\dfrac{l}{h}$. The angle change in the length ED, by Eq. 12, is $\phi_{ED} = -\dfrac{A_w\,l}{(189.0)\,A_w\,h}\,(10^{-3})(87.13 - 24.2) = -\,0.333\,(10^{-3})\,\dfrac{l}{h}$. Then the total angle change in one half of the span length is $\phi_{EC} = 0.018\,(10^{-3})\,\dfrac{l}{h}$. Since ϕ_{EC} is positive, the values of the moment and of the strain at the end of the beam have been underestimated.

When the procedure is repeated in trial 2, assuming $\epsilon_E = 23\,(10^{-3})$, the resultant angle change is $\phi_{EC} = -\,0.058\,(10^{-3})\,\dfrac{l}{h}$. The true value of ϵ_E can

be found by linear interpolation, assuming that ϵ_E varies in proportion to ϕ_{EC}. Thus, $\epsilon_E = \left(22 + \frac{18}{18 + 58}\right) 10^{-3} = 22.25\ (10^{-3})$.

The third trial made with this value of ϵ_E yields $\phi_{EC} = -\ 0.002\ (10^{-3})\ \frac{l}{h}$. Since this is negligible, the value $\epsilon_E = 22.25\ (10^{-3})$ may be considered as the true value. The corresponding value of the m-function can be found by interpolation in the design Table 1(a): $m_E = 52.06 + \frac{0.25}{2}\ (53.33 - 52.06) = 52.22$ (kips/in.^{2}); and

$$M = 26.11\ (\text{kips/in.}^2)\ A_w\, h = C_M\, A_w\, h \qquad (42a)$$

in which $C_M = 26.11$ kips per sq in., is a coefficient. Linear interpolation has been found to work quite well, when the requisite statically unknown strain extends, as in this particular case, into the strain-hardening region. Under these circumstances, two reasonably close trials are sufficient for the solution. When the unknown strain is found in the plastic range, its variation with ϕ is not linear, and more than two trials may be needed for accurate results. In that case ϵ is very unstable.

The required intensity of the uniform loading (w) is now found by statics: $(26.11 + 22)\ A_w\, h = \frac{1}{8}\, w\, l^2$ and

$$w = 384.9\ (\text{kips/in.}^2)\ \frac{A_w\, h}{l^2} = C_w\, \frac{A_w\, h}{l^2} \qquad (42b)$$

in which $C_w = 384.9$ kips per sq in. is a coefficient.

The deflection δ at the center of the beam is found, by reference to Fig. 23, as the sum of two deflections caused by deformations of the parts ED and DC: $\overline{DC} = 0.5\, l \sqrt{\frac{22 + 22}{26.11 + 22}} = 0.4782\ l$; and $\overline{ED} = 0.0218\, l$.

The position of point F, Fig. 23, where the straight line ed cuts the axis, is found from $\bar{x} = 0.5\, l = \frac{26.11}{26.11 - 22}\ (0.0218\ l) = 0.3614\, l$.

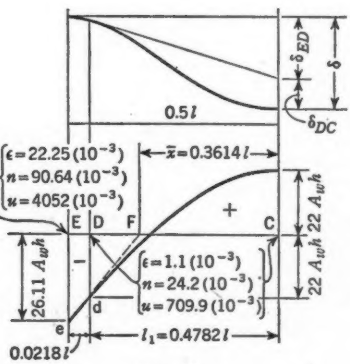


FIG. 23

The shearing force on the length ED is

$$V = \frac{4.11\, A_w\, h}{0.0218\, l} = 188.5\ (\text{kips/in.}^2)\ \frac{A_w\, h}{l} = C_V\, \frac{A_w\, h}{l} \qquad (42c)$$

The values of n and u at point E are found from Table 1(a) by interpolation. They are stated in Fig. 23 together with the values of the same functions at point D. By Eq. 21a, $\delta_{ED} = -\left[\frac{h}{2}\left(\frac{1}{188.5}\frac{l}{h}\right)^2 (4{,}052 - 709.9)\ (10^{-3})\right.$

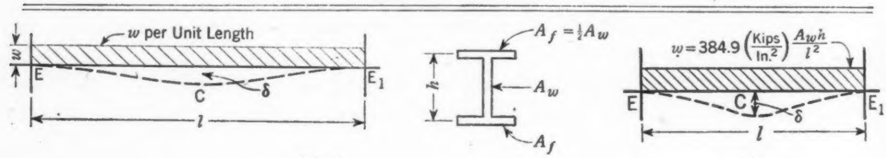
$+ \frac{1}{188.5} \frac{l}{h} (0.3614\, l) (90.64 - 24.2) 10^{-3} \Big] = - 0.1743 (10^{-3}) \frac{l^2}{h}$. By elastic relations, $\delta_{DC} = - (\tfrac{2}{3}) \frac{2}{h} (1.1) 10^{-3} (0.4782\, l) (\tfrac{3}{8}) (0.4782\, l) + \frac{1.1}{h} (10^{-3}) (0.4782\, l) \times \tfrac{1}{2} \times (0.4782\, l) = 0$. Therefore,

$$\delta = - 0.1743 (10^{-3}) \frac{l^2}{h} = - C_\delta \frac{l^2}{h} \ldots\ldots\ldots\ldots\ldots (42d)$$

The results are summarized in line 3, Table 2(*a*) under the title "Inelastic Condition." For comparison, two other sets of capacity loads and stresses are also given, one determined by the theory of limit design (line 2) and the other by the elastic theory (line 1), corresponding to the condition when the yield

TABLE 2.—DEFLECTIONS, STRAINS, STRESSES, AND MOMENTS UNDER CAPACITY LOADINGS

(Units Are: σ C_W, and C_M, Kips per Square Inch; and ϵ and C_δ, Inches per Inch)



(*a*) (*b*)

Line	Description	C_w (Eq. 42*b*)	1,000 C_δ (Eq. 42*d*)	(*a*) CENTER, POINT C 1,000 ϵ	σ	C_M (Eq. 42*a*)	(*b*) END, POINT E 1,000 ϵ	σ	C_M (Eq. 42*a*)
	(*a*) PRINCIPAL QUANTITIES								
1	Limiting elastic condition	264	0.0687	0.55	16.5	+11	1.1	33	−22
2	Limit design	352	..	..	33	+22	..	33	−22
3	Inelastic condition	384.9	0.1743	1.1	33	+22	22.25	35.49	−26.11
	(*b*) RESIDUAL QUANTITIES								
4	Loading	384.9	0.1743	+1.1	+33	+22	+22.25	+35.49	−26.11
5	Unloading	384.9	−0.1001	−0.80	−24.05	−16.04	− 1.60	−48.10	+32.07
6	Residual condition	384.9	0.0742	+0.30	+ 8.95	+ 5.96	+20.65	− 12.61	+ 5.96

point has first been reached in the beam. In computing these additional data, the following values of the beam constants have been used: E = 30,000 kips per sq in. and

$$I = \tfrac{1}{12} A_w h^2 + 2 A_f \left(\frac{h}{2}\right)^2 = \tfrac{1}{3} A_w h^2 \ldots\ldots\ldots\ldots\ldots (43)$$

since $\frac{A_f}{\frac{1}{2} A_w} = \kappa = 1$.

The residual state of stress (RS) is determined by adding to the moments, stresses, and strains determined in Table 2(*a*) the corresponding functions found by the elastic application of the same loads in the opposite direction.

This is done in Table 2(*b*). The functions added to the original ones are stated in line 5. They are found by increasing the values pertaining to "Limiting Elastic Condition" in Table 2(*a*) in the ratio of the loads, and changing the signs. The negative sign of stresses and strains in Table 2(*b*) indicates an action opposite to the one created by the original loading.

15. Example 2

Given a fixed-ended I-beam with $\kappa = 1$, loaded with a concentrated load P at one eighth of its span length (see Fig. 24), find the value of the load for which the moment under the load just reaches the yield point. Find also the corresponding stresses and strains.

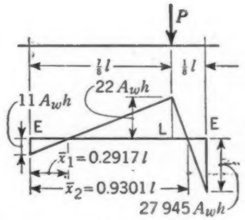


Fig. 24

The conditions in this problem are purposely assumed so as to create a great disparity in the moments at the three critical points—that is, at the two ends and at the load. The structure is twice statically indeterminate. For this reason, the solution involves guessing at two moments or corresponding strains at once, and necessitates at least three trial assumptions instead of the two in the first problem. The moment and strain under the load are known, and the strains at the ends are assumed, using some judgment and bearing in mind the relation between the moments in the elastic range. With the conditions of strain thus known, the angle change on the entire length of the beam and the deflection of end E in relation to the tangent at end E_1 are determined and compared to zero, which value they should possess to satisfy the conditions of restraint. These deformations are found by the double application of Eqs. 12 and 21*c* in the same manner as in Example 1. The quantities entering these formulas are mostly taken from Table 1(*a*) or are computed by geometry in Fig. 24. They are all conveniently arranged in Table 3. The values of the

TABLE 3.—Trial 1 (See Fig. 24)

Quantity	Point E		Point L		Point E_1
$1{,}000\,\epsilon$	0.55		1.1		28
m (kips/in.^{2})	22		44		55.89
$M \frac{1}{A_w h} \left(\frac{\text{kips}}{\text{in.}^2}\right)$	11		22		27.945
$1{,}000\,n$ (kips/in.^{2})	6.05		24.2		182.9
$1{,}000\,u$ (kips^2/in.^{4})	88.74		709.9		9,045
$1{,}000\,(n_0 - n_{01})$ (kips/in.^{2})		18.15		158.7	
$1{,}000\,(u_0 + u_{01})$ (kips^2/in.^{4})		798.64		9,754.9	
$V \frac{l}{A_w h}$ (kips/in.^{2})		37.72		399.6	
$\bar{x}(1/l)$		0.2917		0.9301	

strain functions below the proportional limit are determined by comparison with the values at the proportional limit, where $\epsilon = 1.1\,(10^{-3})$, by the following relations evident from Eqs. 26: m is proportional to ϵ, n is proportional to ϵ^2, and u is proportional to ϵ^3.

Trial 1.—From values assembled in Table 3: $\phi = \left(\frac{18.15}{37.72} - \frac{158.7}{399.6}\right) 10^{-3} \frac{l}{h}$ $= + 0.0836\ (10^{-3})\frac{l}{h}$; and $\delta = \left\{\left[\frac{1}{2}\frac{798.64}{(37.72)^2} + \frac{18.15}{37.72}(0.2917)\right] - \left[\frac{1}{2}\frac{9{,}754.9}{(399.6)^2} + \frac{158.7}{399.6}(0.9301)\right]\right\} 10^{-3}\frac{l^2}{h} = + 0.0211\ (10^{-3})\frac{l^2}{h}$.

TABLE 4.—RESULT OF SUCCESSIVE TRIALS, EXAMPLE 2

Trial	$10^3\,\epsilon_E$	$10^3\,\epsilon_L$	$10^3\,\epsilon_{E1}$	$C\varphi = \varphi(10)^3\,h/l$	$C\delta = \delta(10)^3\,h/l^2$
(1)	(2)	(3)	(4)	(5)	(6)
One....	0.55	1.1	28	+0.0836	+0.0211
Two....	0.77	1.1	28	−0.1085	−0.0346
Three..	0.55	1.1	30	−0.0034	−0.0650

Two other similar trials are now made, changing the values of the assumed unknown strains ϵ_E and ϵ_{E1}, one at a time. The corresponding values of ϕ and δ are computed, and the results are given in Table 4.

Assuming a linear variation of ϕ and δ with ϵ_E and ϵ_{E1} in the vicinity of zero, the rates of variation are:

$$\frac{\partial\phi}{\partial\epsilon_E} = \frac{-0.1085 - 0.0836}{0.77 - 0.55}\frac{l}{h} = -0.873\frac{l}{h} \qquad (44a)$$

$$\frac{\partial\delta}{\partial\epsilon_E} = \frac{-0.0346 - 0.0211}{0.77 - 0.55}\frac{l^2}{h} = -0.253\frac{l^2}{h} \qquad (44b)$$

$$\frac{\partial\phi}{\partial\epsilon_{E1}} = \frac{-0.0034 - 0.0836}{30 - 28}\frac{l}{h} = -0.0435\frac{l}{h} \qquad (44c)$$

and

$$\frac{\partial\delta}{\partial\epsilon_{E1}} = \frac{-0.0650 - 0.0211}{30 - 28}\frac{l^2}{h} = -0.0430\frac{l^2}{h} \qquad (44d)$$

The unknown increments $d\epsilon_E$ and $d\epsilon_{E1}$, which must be added to the values of the unknowns ϵ_E and ϵ_{E1}, assumed in the first trial, to make ϕ and δ both equal to zero, are found from the two simultaneous equations:

$$-0.873\,d\epsilon_E - 0.0435\,d\epsilon_{E1} + 0.0836\,(10^{-3}) = 0$$
$$-0.253\,d\epsilon_E - 0.0430\,d\epsilon_{E1} + 0.0211\,(10^{-3}) = 0$$

From these: $d\epsilon_E = 0.101\ (10^{-3})$; and $d\epsilon_{E1} = -0.108\ (10^{-3})$. Then the required strains are $\epsilon_E = (0.55 + 0.101)10^{-3} = 0.65\ (10^{-3})$; and $\epsilon_{E1} = (28 - 0.108)\ 10^{-3} = 27.9\ (10^{-3})$.

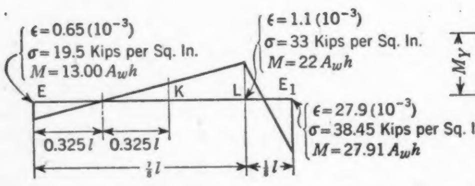


FIG. 25

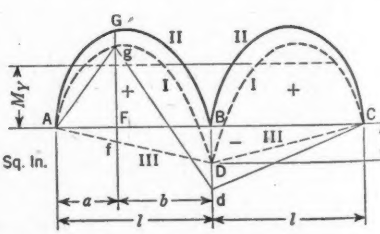


FIG. 26

The values of stresses and moments (Fig. 25) are now found by interpolation from Table 1(*a*). The load *P* equals the sum of the shearing forces in the two

parts of the beam:

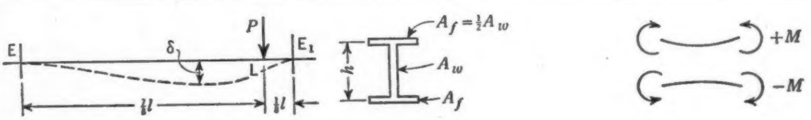
$$P \doteq \left(\frac{13.00 + 22.00}{0.875} + \frac{22.00 + 27.91}{0.125} \right) \frac{A_w h}{l}$$

$$= 439.32 \left(\frac{\text{kips}}{\text{in.}^2} \right) \frac{A_w h}{l} = C_P \frac{A_w h}{l} \quad \ldots\ldots\ldots\ldots (45a)$$

The maximum deflection δ occurs in the elastic region of the beam at point

TABLE 5.—Deflections, Strains, Stresses, Moments Under Capacity Loadings

Load	Description
1	Limiting elastic condition in which P is such that the yield point is first reached in the beam.
2	Limit design.
3	Inelastic condition 1; beam loaded as in Example 2.
4	Inelastic condition 1; residual quantities after removal of load.
5	Inelastic condition 2; beam loaded, and the yield point is just reached at the far end E.
6	Inelastic condition 2; residual quantities.



Load (description above)	$C_P = P\frac{l}{A_w h}$ (kips/in.²)	$C\delta = \delta(10)^3 \frac{h}{l^2}$	(a) End E: 1,000 ϵ	(a) End E: σ (kips/in.²)	$C_M = M\frac{1}{A_w h}$ (kips/in.²)	(b) Point L: 1,000 ϵ	(b) Point L: σ (kips/in.²)	(b) Point L: $C_M = M\frac{1}{A_w h}$ (kips/in.²)	(c) End E: 1,000 ϵ	(c) End E: σ (kips/in.²)	(c) End E: $C_M = M\frac{1}{A_w h}$ (kips/in.²)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
1.....	248.72	0.0230	+0.157	+4.71	−3.14	+0.275	+8.125	+5.5	+1.1	+33	−22
2.....	402.29	..	..	+33	−22	..	+33	+22	..	+33	−22
3.....	439.32	0.0915	+0.65	+19.5	−13.00	+1.1	+33	+22	+27.9	+38.45	−27.91
4.....	0	0.0515	+0.35	+10.5	−7.00	+0.575	+17.25	+11.5	+25.8	−24.55	+14.1
5.....	554.48	0.2313	+1.1	+33	−22	+23.5	+36.18	+26.51	+62	+50.06	−35.87
6.....	0	0.1905	+0.72	+21.63	−14.42	+22.84	+16.26	+13.23	+59.35	−29.54	+17.19

K, Fig. 25, where the slope is zero. It is found by the elastic formula:

$$\delta = -2\,(0.65) \times (10^{-3}) \times \tfrac{1}{2}\,(0.325)\,\tfrac{2}{3}\,(0.65)\,\frac{l^2}{h} = -0.0915\,(10^{-3})\,\frac{l^2}{h} \ldots (45b)$$

These results are incorporated in Table 5, together with the conditions of loading, given for comparison.

16. Example 3

Given a two-span continuous I-beam (Fig. 26), with $\kappa = 1$ acted on by a movable concentrated load; find the value of the load for which the stress at

the support just reaches the yield point. It is assumed that the passage of the load producing this stress condition is preceded by several passages of smaller loads, whose values increase gradually to the capacity value.

A concentrated load passing over the beam creates certain maximum positive moments under the load; and in numerical value these moments exceed the negative moments produced at the middle support. Therefore, under the conditions of the problem, the inelastic deformations occur only under the loads. Curve I in Fig. 26 represents the curve of maximum positive moments. The middle sections of both spans, on which the moments exceed the yield point value $M_Y = 22\, A_w\, h$, undergo inelastic deformations, making the beam concave upward. When the load is removed, the residual deformations demand the presence of downward reactions at the outer supports to preserve the conditions of restraint. Therefore, the residual moment condition is such that it can be represented by the line III, Fig. 26. The final location of curve III has been reached gradually in the course of several previous passages of the increasing load P, and it will not be displaced farther, on a new passage, as long as the load P remains constant.

Curve II, Fig. 26, represents the curve of positive moments existing under the load, caused by the elastic application of the load P. When load P arrives at an arbitrary point F, the actual moment diagram is AgdC, Fig. 26. By the principle expressed in Eq. 41, this diagram differs from the diagram caused by the elastic application of the same load, by the ordinates of the curve III. Therefore, $\overline{\text{Gg}} = \overline{\text{fF}}$, and $\overline{\text{Dd}}$ is the moment produced by the elastic application of the load. The value of the intercept Dd, Fig. 26, varies depending on the location of the load. Since the intercept BD is constant, the greatest negative bending moment Bd occurs at the same position of the load as in the elastic range, and by the conditions of the problem it is equal to 22 (kips/in.^{2}) $A_w\, h$.

The solution of this problem involves the use of the elastic formulas for the moments and consists of the following operations:

a. The unknown load P is assumed;
b. Using this value of P, the maximum elastic ordinate dD and the ordinates of the curves II and III are computed;
c. Curve I is found by combining the curves II and III;
d. Residual deflection of end A, Fig. 26, in relation to the tangent at the central support B, caused by the moments of curve I is computed by Table 1(*b*); and
e. The reaction necessary to bring end A back is computed, and the moments produced by it are compared with curve III, to check the assumed value of the load P.

In computing the deflection, curve I is replaced by a polygon. Although this curve does not represent a moment diagram for any single position of the load, it must be treated as if it were one. The slopes of its polygonal sides must be used as the shearing forces in the corresponding formulas for the deflection.

The elastic formulas are as follows (Fig. 26):

$$M_B\ (\max) = 0.0961\ P\ l \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(46a)$$

and

$$M_f = \frac{P\ a\ b}{4\ l^3}\left[4\ l^2 - a(l + a)\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(46b)$$

Trial 1.—In Fig. 26, let $P = 136 \frac{\text{kips}}{\text{in.}^2} \times \frac{A_w h}{l}$. Then, by Eqs. 46, the elastic moment, M_B (max) = 13.08 (kips/in.²) $A_w h$; the residual moment at point B is $M_r = (22 - 13.08)\ A_w\ h = 8.92$ (kips/in.²) $A_w\ h$; and the moments of curve III, Fig. 26, are $M_{III} = -\frac{a}{l}$ (8.92) (kips/in.²) $A_w h$. In Table 6, the ordinates of curve I represent the difference between curves II and III. Curve I, slightly simplified, is shown in Fig. 27. The ordinates smaller than 22 $A_w h$ are immaterial, since they do not produce inelastic deformations. All quantities needed in computation are conveniently arranged in Table 7. The data are taken from Table 1(*b*) and Fig. 27.

TABLE 6.—COEFFICIENTS $C_M = \frac{M}{A_w h}$, IN KIPS PER SQUARE INCH, FOR FINDING THE ORDINATES OF CURVE I (See Fig. 26)

a/l	Curve II	Curve III	Curve I
0.266	24.37	−2.37	22.00
0.3	25.77	−2.68	23.09
0.35	27.31	−3.12	24.19
0.39	27.96	−3.48	24.48
0.40	28.09	−3.57	24.52
0.41	28.15	−3.66	24.49
0.45	28.17	−4.01	24.16
0.50	27.61	−4.46	23.15
0.537	26.79	−4.79	22.00

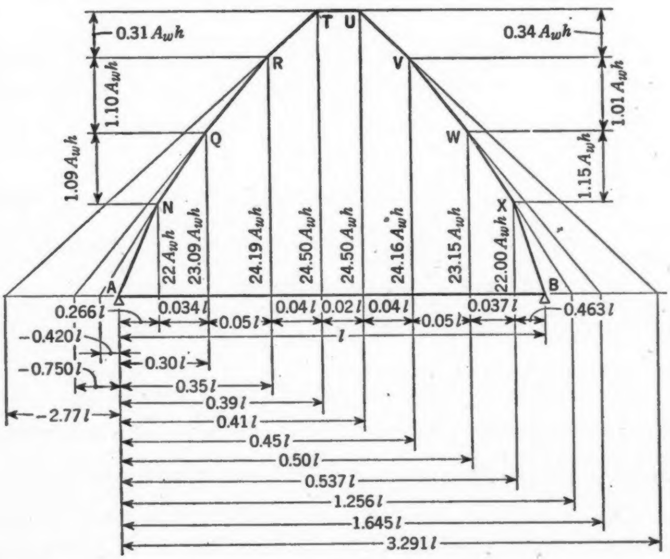


FIG. 27

The residual deflection of end A, Fig. 26, in relation to the tangent at middle support B is determined by applying Eqs. 21*a*, 21*b*, and 22*a* to the residual conditions. Eq. 22*a* is used for the interval TU, Fig. 27, where the strain is

TABLE 7.—DEFORMATION DATA (SEE FIG. 27)

Point	$C_M = \frac{M}{A_w h}$ (kips/in.²)	m (kips/in.²)	1,000 ϵ	1,000 n_r (kips/in.²)	1,000 u_r (kips²/in.⁴)	1,000 $(n_{r0} - n_{r1})$ (kips/in.²)	1,000 $(\overline{u_{r0} - u_{r1}})$ (kips²/in.⁴)	1,000 ϵ_r	$C_V = V \frac{l}{A_w h}$ (kips/in.²)	$\frac{\bar{x}}{l}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
N	22.00	44.00	1.1	0	0					
						0.245	11.18	..	32.04	−0.420
Q	23.09	46.18	1.418	0.245	11.18					
						1.323	62.90	..	22.00	−0.750
R	24.19	48.38	2.48	1.568	74.08					
						1.006	48.93	..	7.75	−2.770
T	24.50	49.00	3.6	2.574	123.01					
						0	0	2.376	0	..
U	24.50	49.00	3.6	2.574	123.01					
						1.094	53.21	..	8.5	3.291
V	24.16	48.32	2.4	1.480	69.80					
						1.200	56.99	..	20.2	1.645
W	23.15	46.30	1.445	0.280	12.81					
						0.280	12.81	..	31.1	1.256
X	22.00	44.00	1.1	0	0					

constant. The residual upward deflection is:

$$\delta = 0.1994\ (10^{-3})\ \frac{l^2}{h} \quad \text{.....................} \quad (47)$$

Now apply, elastically, a downward reaction at end A, to bring this end back to its original position. The maximum strain ϵ_e at point B, so induced, is found from the equation $2 \frac{\epsilon_e}{h} \frac{1}{2} l \frac{2}{3} l = 0.1994\ (10^{-3})\ \frac{l^2}{h}$ as follows: $\epsilon_e = 0.2992\ (10^{-3})$.

The residual moment M_r at support B is now found by proportion: $M_r = \frac{0.2992}{1.1}\ 22\ A_w\, h = 5.98$ (kips/in.²) $A_w\, h$, instead of 8.92 $A_w\, h$—a discrepancy of $-2.94\ A_w\, h$. Load P has been assumed too small.

Trial 2.—Assume $P = 136.7$ (kips/in.²) $\frac{A_w\, h}{l}$. Repeating the procedure of trial 1, the discrepancy in moments M_r proves to be $+0.40\ \frac{\text{kips}}{\text{in.}^2} \times A_w\, h$, showing that the assumed value of P exceeds the true value. The difference, however, is undoubtedly less than $0.1\ \frac{A_w\, h}{l}$, and a new trial is unnecessary.

A comparison of this solution with the results obtained by the method of limit design and with the results of the limiting elastic condition is given in Table 8.

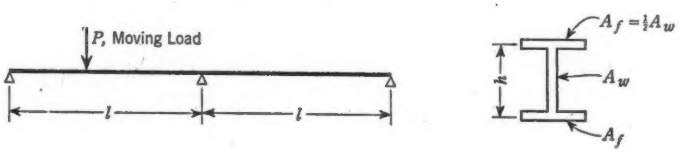
Some error is involved in this solution because an inscribed polygon has been substituted for the curved line of maximum moments. However, even if the residual deflection found on the basis of this polygonal line may be some-

what in error, the discrepancy in P should be negligible, because only a small change in P is needed to affect the deflection appreciably.

If the conditions of the problem are modified in such a way that the load P is assumed to possess its full capacity value on the first passage, rather than to

TABLE 8.—Comparison of Results, Example 3

(Units of C_P, C_M and σ Are in Kips/In.^{2})



Description	$C_P = P \dfrac{l}{A_w h}$ $\left(\dfrac{\text{kips}}{\text{in.}^2}\right)$	(a) Under Load			(b) At Center Support		
		1,000 ϵ	σ (kips/in.^{2})	$C_M = M \dfrac{1}{A_w h}$ $\left(\dfrac{\text{kips}}{\text{in.}^2}\right)$	1,000 ϵ	σ (kips/in.^{2})	$C_M = M \dfrac{1}{A_w h}$ $\left(\dfrac{\text{kips}}{\text{in.}^2}\right)$
Limiting elastic condition	105.9	1.1	33	22	0.51	15.27	10.18
Limit design	126.1	..	33	22	..	33	22
Inelastic condition	136.7	5.9	33	24.68	1.1	33	22

be built up gradually in the course of previous passages, the problem becomes very laborious, although possible of solution. The magnitude of the capacity load found under those circumstances will be smaller.

17. Limit Design

Consider a statically indeterminate structure subjected to flexure, such as one of the beams analyzed in the foregoing examples, and let the loading acting on it increase gradually in intensity. The increase in loading is accompanied by an increase in stresses, and several sections of the beam, one by one, reach the yield point. When the number of such sections at and beyond the yield point, referred to as critical sections, is one larger than the number of static unknowns, the capacity loading is reached, since a small further increase in intensity leads quickly to a large deformation.

In his theory of limit design Professor Van den Broek proposes that the capacity loading be found by assuming that the values of the bending moments at all the critical sections are equal to the value for which the yield stress is just reached. This assumption underestimates, somewhat, the moments at the critical sections reached earlier; but on the whole it is not far from truth, and (what is important) it permits finding the capacity loading by statics alone.

It is conceivable, however, that before the structure yields at the last critical section it may fail at the first critical section, and this vital point must be cleared before the theory of limit design can be accepted. This question can

be answered only by an exact theory of bending, such as the one presented in this paper.

In the foregoing examples, the true capacity loadings, together with the accompanying states of stress, have been determined and compared with the limiting elastic loadings and the loadings found by the limit design. Some features of the mechanical behavior of structures, made apparent by these examples, are worthy of notice.

The moment at the first critical section does not remain constant, as the moment at the second section approaches yielding, but it increases. This increase, although relatively slow compared to the elastic stage, must still be sufficient to produce the necessary angle change. In the case of stationary loads, as in Examples 1 and 2, Sections 14 and 15, the strain at the first critical section must almost inevitably extend into the strain-hardening region. This fact can be best illustrated by reference to Fig. 22, assuming, for simplicity, an I-beam consisting mostly of flanges with practically no web. As the positive moment increases, the positive angle change increases also, demanding (by the conditions of restraint) an equal negative angle change at the support; but, as long as the flange stress at the end of the beam remains within the plastic range, the end moment M_E must remain constant, and virtually no increase in the negative angle change is possible, because all yielding is concentrated on an infinitesimal length near the end. If the stress-strain curve of the material of the beam contains no rising part corresponding to strain hardening, the beam must fail at the ends under a load no larger than the one which causes first yielding.

The presence of the web in an I-beam does not modify this condition appreciably, as may be noted from Eq. 13*b* and Table 1(*a*). Eq. 13*b* shows that the angle change occurring on a constant length x_0 of a beam subjected to a triangular moment (Fig. 7) varies as the ratio $\frac{n_0}{m_0}$ corresponding to the stress condition at the end; and this ratio, according to Table 1(*a*), remains nearly constant in the later part of the plastic range, reaching a limit which in case of $\kappa = 1$ is only some 30% higher than the value at the end of the elastic part. In the strain-hardening region, the ratio $\frac{n_0}{m_0}$ again increases quickly, so that its value, required by the conditions of deformation, is soon reached.

An important conclusion can be drawn from this discussion: It is not the plastic part of the stress-strain curve, or at least not this part alone but mostly the strain-hardening part, that makes possible a favorable distribution of moments in the region beyond the elastic stage, as visualized in the theory of limit design.

The steepness and length of the strain-hardening curve, characterized roughly by the ratio of the ultimate stress and the yield-point stress and by the percentage elongation at failure, are very significant. In mild steel these factors are quite favorable, and the strains at the points of critical moments, as a rule, do not extend beyond the very beginning of the strain-hardening range. Even in the extreme case of Example 2, characterized by a great disparity of

critical moments in the elastic stage, the strain at the first critical section only reaches the value of 6.2%, which is far short of failure, when the third critical section has just entered the yielding stage. The result may be quite different, however, with some alloy steels and nonferrous metals, possessing a much smaller ratio of the ultimate stress to yield-point stress than the mild steel.

In Example 3, dealing with a movable load, the region of inelastic strain as distinguished from that in the two other examples, is spread over a large length of the span; and, as a result, an ample angle change, relieving the moments in the middle parts of the spans, is produced by only a moderate extension of strain into the plastic region.

Comparison of the capacity loadings found by different methods in the three examples shows that the limit-design values are always on the safe side, which is the direct result of underestimating the moment carried by the beam at the first critical point. This margin of difference is larger with the stationary loads than with the movable loads, and in beams with $\kappa = 1$ it seems to be usually of the order of from 7% to 10%.

The only condition that may require a special investigation in this connection is the case cited in Section 16 when the moving loads are considered to have their full values on the first passage. Limit design can offer no answer to this problem, other than the one assuming a gradual pre-stressing of the beam by smaller loads—an assumption tending to overestimate the capacity load and to make the design unsafe. This tendency is compensated for, however, by an opposite tendency, as explained; and, when appreciable stationary loads occur at the same time as moving loads, the limit design is probably always safe.

Findings of the exact theory are thus very favorable to the theory of limit design when it is applied to mild steel and when the instability failure is not impending.

It is unfortunate that continuous, or fixed-ended, steel beams are rather rare in building construction. This condition limits the practical field of applicability of the flexural theory of limit design. On the other hand, the possibility of the use of this theory in the design of beams provided with the usual web and flange-angle connections, assumed ordinarily in design as simply supported, seems to be worthy of serious study. If the connections of such beams are sufficiently strong, although quite deformable, the end moments will be brought into action eventually, after some yielding at the center, and the design on the basis of end restraints may prove economical, as well as safe. Such an investigation would have to be based on the flexural theory presented herein and on the experimental data pertaining to the strength and rigidity of the end connections.

18. Conclusions

The material presented in this paper warrants the following conclusions:

1. The proposed flexural theory provides (within the scope of its assumptions) a rigorous method of determining a complete picture of stress and deformation in statically indeterminate or determinate I-beams and channel

beams, irrespective of whether or not the material of the beam obeys Hooke's law. Both the loaded state and the residual state, remaining after the removal of the load, are susceptible to analysis.

2. Favorable redistribution of bending moments, occurring beyond the yield point and taken advantage of in the theory of limit design, is made possible partly by the yielding characteristic of steel, but mostly by its strain-hardening characteristic.

3. Flexural structures of mild steel will not fail at the point of the greatest stress before the capacity load (characterized by approach of large deflections) is reached.

4. Barring the possibility of a failure caused by instability, the value of the capacity load of a flexural structure, found by limit design is on the safe side, with perhaps some rare exceptions when moving loads are involved.

5. Limit design may prove unsafe when applied to alloy steels and non-ferrous metals in which the ratio of the ultimate stress to the yield-point stress is comparatively low.

6. The question of applying limit design to steel beams provided with the usual end connections is worthy of study, with a view to placing reliance in design on the end restraints of such beams.

APPENDIX. NOTATION

The following letter symbols, adopted for use in this paper and for the guidance of discussers, conform essentially to American Standard Letter Symbols for Mechanics (ASA–Z10.3–1942), prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1942. In general, a subscript C denotes "center"; E denotes "end"; e denotes "elastic"; f denotes "flange"; r denotes "residual"; u denotes "ultimate"; w denotes "web"; and Y denotes "yield."

A = area of cross section;

b = breadth, or width;

C = a coefficient with various subscripts before the expressions of different functions determined in the examples of Sections 14, 15, and 16:

$$C_M = M \times \frac{1}{A_w h} \text{ (Eq. 42}a\text{)}; \qquad C_P = P \times \frac{l}{A_w h} \text{ (Eq. 45}a\text{)};$$

$$C_V = V \times \frac{l}{A_w h} \text{ (Eq. 42}c\text{)}; \qquad C_w = w \times \frac{l^2}{A_w h} \text{ (Eq. 42}b\text{)};$$

$$C_\delta = \delta \times \frac{h}{l^2} \text{ (Eq. 42}d\text{)}; \qquad C_\varphi = \phi \times \frac{h}{l}$$

d = distance or length; d in Fig. 7 is the distance from the point of zero moment, point A;

E = modulus of elasticity;
h = height, or depth;
l = span length of beam;
M = bending moment;
m = derived moment function of the material of the beam for an I-beam, Eq. 4;
m_1 = derived moment function of the material of the beam for a rectangular beam, Eq. 1;
n = derived slope function of the material of an I-beam, Eq. 10;
P = concentrated load;
q = derived shear function of the material of an I-beam, Eq. 6*b*;
u = derived deflection function of the material of an I-beam, Eq. 17*b*;
V = total shear;
w = load per unit length of span;
x = distances along the longitudinal axis of a beam; dx = an element of length x;
y = distances from the neutral axis;
δ = deflection;
ϵ = unit strain;
κ = ratio $A_f/\frac{1}{2}A_w$;
σ = unit bending and direct stress;
τ = unit shear stress; and
ϕ = angle change.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

BEAM DEFLECTIONS BY SECOND AND THIRD MOMENTS

BY HSU SHIH-CHANG,[1] ESQ.

SYNOPSIS

The various methods for computing beam deflections may be classified into two main groups—the double-integration method and the moment-area method. In both methods, beam deflection is considered a function of bending moment; and bending moment, in turn, is a function of the force system acting on the beam. In other words, deflection is computed, not directly from the forces acting on the beam, but rather indirectly through the medium of bending moment. Such indirect approach naturally makes the computations lengthy and complicated, and the aim of this paper is to suggest a more direct solution.

INTRODUCTION

With the conception of second and third moments of forces and couples, formulas are derived expressing slope deviation as a function of the second moment of forces and couples, and deflection as a function of the second and third moments of forces and couples. Although the fundamental differential equation of beam deflection—

$$\frac{d^2y}{dx^2} = \frac{M}{E\,I} \qquad (1)$$

—must be used in the derivation of such formulas, the final results are free from expressions of bending moment and they have the unique features of simplicity and lucidity.

The method can also be applied to a beam of variable moment of inertia provided that it is segmentally constant and provided that the secondary effects of axial thrust or tension, as well as the effect of the deflection due to shear, are not considered.

Notation.—The letter symbols in this paper are defined where they first appear, in the text or by illustrations, and are assembled alphabetically in the Appendix for convenience of reference.

NOTE.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by **August 1, 1947.**

[1] Graduate, College of Eng., National Chung Cheng Univ., Nanchang, Kiangsi, China.

Second and Third Moments of Forces and Couples

Definition of Second and Third Moments.—Unless otherwise modified for a given case, this paper is based on the following assumptions: All forces are perpendicular to the reference line (ON, Figs. 1 and 2), all couples are considered to act at some point on the reference line, and the moment center is always taken on the reference line.

The second moment M'' and the third moment M''' of any force P_A, with respect to any point N taken as the moment center, are defined as the force P_A multiplied by the square and cube, respectively, of the perpendicular distance from the force to point N. Thus (see Fig. 1):

$$M_N = P_A\, a_A; \qquad M''_N = P_A\, a^2_A; \text{ and } M'''_N = P_A\, a^3{}_A \ldots\ldots (2)$$

in which the point force P_A is considered positive when acting upward and the moment arm a_A (Fig. 1) is positive when point A is on the left of point N.

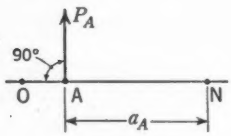


Fig. 1

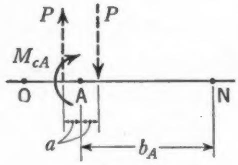


Fig. 2

Fig. 2 shows a couple M_{cA} acting at point A on the reference line ON. This couple can be replaced by two equal and opposite forces:

$$P = \frac{M_{cA}}{2\,a} \ldots\ldots (3)$$

in which the moment arm a is infinitely small. For this case the second moment is written:

$$M''_N = P\,(b_A + a)^2 - P\,(b_A - a)^2 = 2\,(2\,P\,a)\,b_A \ldots\ldots\ldots\ldots (4)$$

In other words,

$$M''_N = 2\,M_{cA}\,b_A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5a)$$

and the third moment, similarly, is written:

$$M'''_N = P\,(b_A + a)^3 - P\,(b_A - a)^3 = 3\,M_{cA}\,b^2{}_A \ldots\ldots\ldots (5b)$$

In Eqs. 5 the couple M_c is positive when it acts in a clockwise direction; and the reference distance, or "moment arm" b_A, is positive when point A is on the left of point N. If the couple M_c acts at point N:

$$M_N = M_{cA};\ M''_N = 0; \text{ and } M'''_N = 0 \ldots\ldots (6)$$

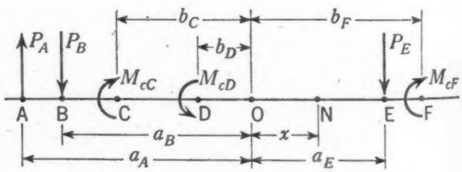


Fig. 3

Characteristics of Second and Third Moments.—Consider the balanced force system shown in Fig. 3, with points O and N, in turn, taken as the moment center. From the conditions of equilibrium:

$$\sum P = P_A - P_B - P_E = 0 \ldots\ldots\ldots\ldots\ldots\ldots (7a)$$

$$M_O = P_A\,a_A - P_B\,a_B + P_E\,a_E + M_{cC} - M_{cD} + M_{cF} = 0 \ldots\ldots (7b)$$

and

$$M_N = P_A\,(a_A + x) - P_B\,(a_B + x) + P_E\,(a_E - x) + M_{cC} - M_{cD} + M_{cF} = 0 \ldots (7c)$$

Taking algebraic signs into consideration, Eqs. 7 can be written in more general terms as follows:

$$\sum_1^n P_n = 0 \ldots\ldots (8a)$$

$$M_O = \sum_1^n (P_n\,a_n) + \sum_1^m M_{cm} = 0 \ldots\ldots (8b)$$

and

$$M_N = \sum_1^n P_n(a_n + x) + \sum_1^m M_{cm} = 0 \ldots\ldots (8c)$$

The second moments of the balanced force system (Fig. 3) with respect to points O and N, respectively, are:

$$M''_O = \sum_1^n (P_n\,a^2_n) + 2\sum_1^m (M_{cm}\,b_m) \ldots\ldots (9a)$$

and

$$M''_N = \sum_1^n P_n(a_n + x)^2 + 2\sum_1^m M_{cm}\,(b_m + x) = \sum_1^n (P_n\,a^2_n) + 2\sum_1^m (M_{cm}\,b_m) + x^2\sum_1^n P_n + 2\,x\left[\sum_1^n (P_n\,a_n) + \sum_1^m M_{cm}\right] \ldots (9b)$$

Substituting Eqs. 8 in Eqs. 9:

$$M''_N = \sum_1^n (P_n\,a^2_n) + 2\sum_1^m (M_{cm}\,b_m) = M''_O \ldots\ldots ((10a)$$

or

$$M'' = \text{a constant} \ldots\ldots (10b)$$

In Eq. 10*b* the subscript for M'' is unnecessary since the location of the moment center is immaterial. Eq. 10*b* defines the first characteristic of the second and third moments, for forces and couples; thus:

Theorem I. The second moment of any balanced force system with respect to any point on the reference line is a constant.

Similar to Eqs. 9, the third moments of the balanced force system (Fig. 3) with respect to points O and N, respectively, are:

$$M'''_O = \sum_1^n (P_n\,a^3_n) + 3\sum_1^m (M_{cm}\,b^2_m) \ldots\ldots (11a)$$

$$M'''_N = \sum_1^n P_n(a_n + x)^3 + 3\sum_1^m M_{cm}\,(b_m + x)^2 = \sum_1^n (P_n\,a^3{}_n)$$

$$+ 3\sum_1^m (M_{cm}\,b^2{}_m) + 3\,x\left[\sum_1^n (P_n\,a^2{}_n) + 2\sum_1^m (M_{cm}\,b_m)\right]$$

$$+ 3\,x^2\left[\sum_1^n (P_n\,a_n) + \sum_1^m M_{cm}\right] + x^3\sum_1^n P_n \quad \text{(11b)}$$

Substituting Eqs. 8 and 10 in Eqs. 11:

$$M'''_N = M'''_O + 3\,M''_O\,x = M'''_O + 3\,M''\,x \quad \text{(12)}$$

Eq. 8 defines the second characteristic of second and third moments:

> Theorem II. The third moment of a balanced force system with respect to any point N on the reference line is equal to the third moment of the force system with respect to any other point O on the reference line, plus three times the distance ON multiplied by the second moment of the force system (with respect to any point on the reference line). The value of ON is positive when point N is on the right side of point O.

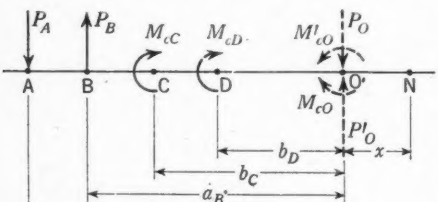


FIG. 4

The next step is to consider the unbalanced force system shown in Fig. 4, consisting of the parallel forces P_A and P_B and couples M_{cC} and M_{cD}. Let the resultant of the unbalanced force system be represented by a force P_O and a couple M_{cO}, both acting at point O; thus:

$$P_O = P_A - P_B \quad \text{(13a)}$$

and

$$M_{cO} = M_{cC} + M_{cD} - P_A\,a_A + P_B\,a_B \quad \text{(13b)}$$

The first, second, and third moments of the given unbalanced force system remain unchanged if two equal and opposite forces (P_O and P'_O) and two equal and opposite couples (M_{cO} and M'_{cO}) are introduced, all acting at point O, as shown by dotted lines in Fig. 4. The given unbalanced force system (P_A, P_B, M_{cC}, and M_{cD}), together with its equilibrant (the force P'_O and the couple M_{cO}) forms a balanced force system. Thus, with any point N on the reference line taken as the moment center: N''_N of the given unbalanced force system (P_A, P_B, M_{cC}, and M_{cD}) is equal to the second moment M''_N of the balanced force system (P_A, P_B, M_{cC}, M_{cD}, P'_O, and M'_{cO}) plus the second moment M''_N of the resultant of the unbalanced force system (P'_O and M_{cO}). Since the second moment of a balanced force system is constant (see theorem I and Eq. 10*b*), the second moment M''_N of the balanced force system (P_A, P_B, M_{cC}, M_{cD}, P'_O, and M'_{cO}) is equal to the second moment M''_O of the same balanced force system, which is equal to the second moment M''_O of the given unbalanced force system plus the second moment M''_O of the equilibrant of the given unbalanced force system (P'_O and M'_{cO}).

From Eq. 6, the second moment M''_O of the equilibrant (P'_O and M'_{cO}) vanishes. Hence:

$$M''_N = M''_O + M''_{O,N} \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

Eq. 14 defines the third characteristic of second and third moments:

Theorem III. The second moment (M''_N) of the unbalanced force system with respect to any point N on the reference line is equal to the second moment of the unbalanced force system (M''_O) with respect to any other point (O), plus the second moment of the resultant of the unbalanced force system ($M''_{O,N}$) at point O (P_O and M_{cO}) with respect to point N.

Similarly, it can be shown that:

$$M'''_N = M'''_O + 3\, M''_O\, x + M'''_{O,N} \ldots\ldots\ldots\ldots (15)$$

which defines theorem IV, as follows:

Theorem IV. The third moment (M'''_N) of the given unbalanced force system (P_A, P_B, M_{cC}, and M_{cD}) with respect to any point N on the reference line is equal to the third moment (M'''_O) of the unbalanced force system with respect to any other point O, also on the reference line, plus three times the product of x and the second moment (M''_O) of the unbalanced force system, with respect to point O; plus the third moment ($M'''_{O,N}$) of the resultant of the unbalanced force system at point O (P_O and M_{cO}) with respect to point N.

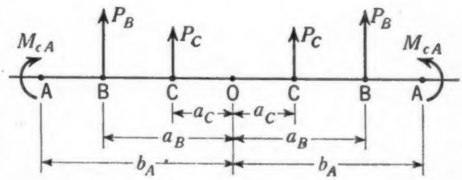


FIG. 5

Finally, for a force system symmetrical about a point O (for example, as shown in Fig. 5), the third moment with respect to point O is:

$$M'''_O = 3\, M_{cA}\, b^2_A + 3\,(-\, M_{cA})\,(-\, b_A)^2 + P_B\, a^3_B + P_B\,(-\, a_B)^3 + P_C\, a^3_C + P_C\,(-\, a_C)^3 = 0 \ldots (16a)$$

or

$$M'''_O = 0 \ldots\ldots\ldots\ldots\ldots\ldots (16b)$$

Eq. 16*b* defines the fifth characteristic:

Theorem V. The third moment of any symmetrical force system with respect to the point of symmetry is equal to zero.

SECOND AND THIRD MOMENTS OF FORCE SYSTEMS ACTING ON STRAIGHT BEAMS

A straight beam may be subjected to any system of loading; but, if the secondary effect of axial thrust or tension is neglected, only the couples and transverse loads applied on the beam axis cause flexure; any forces other than these need not be considered. The following typical cases of loading are particularly important in engineering problems.

Case 1. Concentrated Forces and Couples Acting Indirectly on Beam Axis.— For normal forces and couples acting directly on beam axis, use Eqs. 2 to 5,

taking the beam axis as the reference line. For oblique or indirect loads, replace the given force system by an equivalent force system acting directly on the beam axis, as shown in Fig. 6.

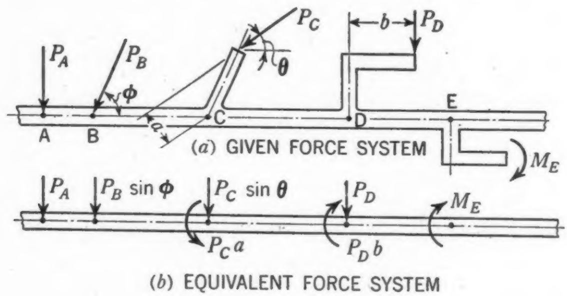


FIG. 6

Case 2. Uniformly Distributed Load Normal to Beam Axis.—By direct integration (see Fig. 7(*a*)):

$$M''_O = \int_{-l/2}^{l/2} \left(- \frac{W}{l}\, dz \right) z^2 = - \frac{W\, l^2}{12} \quad \text{............} (17a)$$

which is numerically equal to the area of the simple beam moment diagram,

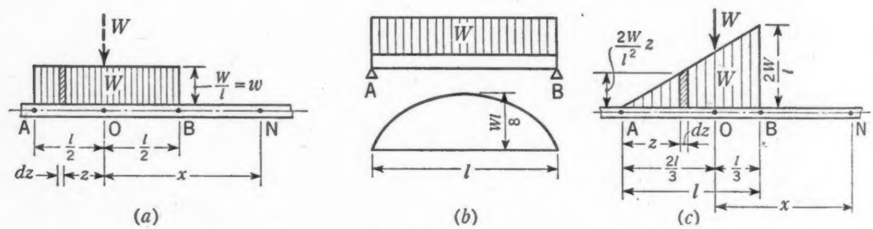


FIG. 7

as shown in Fig. 7(*b*). Since the load is symmetrical about point O (Fig. 7(*a*)), Eq. 16*b* yields:

$$M'''_O = 0 \quad \text{........................} (17b)$$

By Eq. 14:

$$M''_N = M''_O + M''_{O,N} = - \frac{W\, l^2}{12} - W\, x^2 \quad \text{..........} (18a)$$

By Eq. 15:

$$M'''_N = M'''_O + 3\, M''_O\, x + M'''_{O,N} = - \frac{W\, l^2}{4}\, x - W\, x^3 \quad \text{....} (18b)$$

If $x = l/2$:

$$M''_B = - \frac{W\, l^2}{3}; \text{ and } M'''_B = - \frac{W\, l^3}{4} \quad \text{.............} (19)$$

If $x = - l/2$:

$$M''_A = - \frac{W\, l^2}{3}; \text{ and } M'''_A = + \frac{W\, l^3}{4} \quad \text{.............} (20)$$

Case 3. Triangular Load Normal to Beam Axis.—By integration, referring to Fig. 7(*c*):

$$M''_O = \int_0^l \left(- \frac{2\,W}{l^2}\, z\, dz \right) \left(\tfrac{2}{3}\, l - z\right)^2 = - \frac{W\, l^2}{18} \quad \text{(21a)}$$

and

$$M'''_O = \int_0^l \left(- \frac{2\,W}{l^2}\, z\, dz \right) \left(\tfrac{2}{3}\, l - z\right)^3 = - \frac{W\, l^3}{135} \quad \text{(21b)}$$

By Eqs. 14 and 15:

$$M''_N = M''_O + M''_{O,N} = - \frac{W\, l^2}{18} - W\, x^2 \quad \text{(22a)}$$

and

$$M'''_N = M'''_O + 3\, M''_O\, x + M'''_{O,N} = - \frac{W\, l^3}{135} - \frac{W\, l^2}{6}\, x - W\, x^3 \quad \text{(22b)}$$

For points A and B, assume that $x = -\tfrac{2}{3}\, l$ and $x = +\tfrac{1}{3}\, l$, respectively; thus:

$$M''_A = - \frac{W\, l^2}{2}; \text{ and } M'''_A = \tfrac{2}{5}\, W\, l^3 \quad \text{(23a)}$$

and

$$M''_B = - \frac{W\, l^2}{6}; \text{ and } M'''_B = - \frac{W\, l^3}{10} \quad \text{(23b)}$$

Second and third moments for other loading conditions may be computed by integration and substitution into Eqs. 14 and 15.

Method of Second and Third Moments for Computing Beam Deflections

Defining the x-axis as the undeflected position of the beam axis (which is the "reference line" defined in the preceding section) and the y-axis as the plane normal to the x-axis, with positive directions as shown in Fig. 8, let: δ_A be the vertical deflection at any point A, positive when upward; δ_{BA} be equal to $\delta_B - \delta_A$ and equal to the vertical deflection of point B relative to that of point A; y_{BA} be the vertical deflection of point B measured from the tangent to the elastic curve of the deflected beam axis at point A (positive when upward); θ_A be the angular deflection at point A, which can be taken as equal to $\frac{d\delta}{dx}$ at point A, assuming the same sign as the latter; and θ_{BA} be equal $\theta_B - \theta_A$ and equal to the angular deflection at point B relative to that at point A. The sign conventions are illustrated in Fig. 8. It follows that:

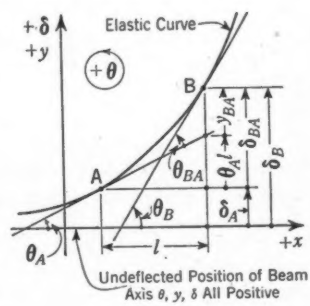


Fig. 8

$$\theta_{BA} = -\,\theta_{AB}; \text{ and } \delta_{BA} = -\,\delta_{AB} \quad \text{(24)}$$

Fig. 9 shows a segment AB of a straight beam with constant flexural rigidity, subject to end shears V_A and V_B; end moments M_A and M_B; transverse loads P_a, P_c, $\cdots$, P_n; and couples M_b and M_d, $\cdots$, M_m.

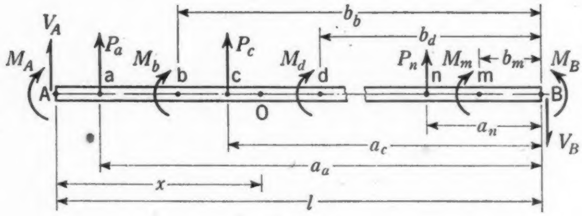


FIG. 9

The well-known differential equation of the elastic curve of the deflected beam axis is

$$\frac{d^2y}{dx^2} = \frac{M}{E\,I} \qquad (25)$$

in which

$$M = M_A + V_A\,x; \quad \text{for} \quad 0 \leqq x \leqq l - a_a \qquad (26a)$$

$$M = M_A + V_A\,x + P_a\,(x - l + a_a); \quad \text{for} \quad l - a_a \leqq x \leqq l - b_b \qquad (26b)$$

$$M = M_A + M_B + V_A\,x + P_a\,(x - l + a_a); \quad \text{for} \quad l - b_b \leqq x \leqq l - a_c \qquad (26c)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$M = M_A + \sum_1^m M_m + V_A\,x + \sum_1^n P_n\,(x - l + a_n); \quad \text{for} \quad l - b_m \leqq x \leqq l \qquad (26d)$$

By integration, the angular deflection of point B relative to point A is:

$$\theta_{BA} = \frac{1}{E\,I}\int_0^l M\,dx = \frac{1}{E\,I}\left\{\left(M_A\int_0^l dx + \sum_1^m M_m \int_{l-b_m} dx\right) + \left[V_A\int_0^l x\,dx + \sum_1^n P_n \int_{l-a_n}^l (x - l + a_n)\,dx\right]\right\} = \frac{1}{2\,E\,I}\,M''_{AB,\,B} \qquad (27)$$

Since the force system shown in Fig. 9 is in equilibrium and the second moment (with respect to any point on the beam axis) is, therefore, a constant:

$$\theta_{BA} = \frac{1}{2\,E\,I}\,M''_{AB} \qquad (28a)$$

in which M''_{AB} is the second moment of the force system acting on the segment AB, including end moments and end shears, and point B is on the right side of point A. By Eq. 24,

$$\theta_{AB} = -\,\theta_{BA} = -\,\frac{M''_{AB}}{2\,E\,I} \qquad (28b)$$

Thus, the more general expression may be written as

$$\theta_{BA} = \pm \frac{M''_{AB}}{2\,E\,I} \qquad (29a)$$

which takes the positive sign when point B is on the right side of point A.

Referring to Fig. 8:

$$\theta_B = \theta_A + \theta_{BA} = \theta_A \pm \frac{1}{2\,E\,I} M''_{AB} \qquad (29b)$$

which takes the positive sign when point B is on the right side of point A.

Eqs. 29 give the first fundamental theorem of the method of second and third moments, for beam deflections:

> Theorem VI. The angle θ_{BA} measured from the tangent to the elastic curve at any point A to the tangent at any other point B on the right side of point A, in a segment AB of a straight beam of constant flexural rigidity, is equal to the second moment of the force system acting on the segment AB, including end moments and end shears of the segment, taken with respect to any point on the undeflected beam axis, divided by twice the flexural rigidity of the beam (that is, by $2\,E\,I$).

The deflection of point B from the tangent at point A (Figs. 8 and 9) is:

$$y_{BA} = \frac{1}{E\,I}\int_0^l\int_0^x M\,dx\,dx = \frac{1}{E\,I}\left\{\left(M_A\int_0^l\int_0^x dx\,dx + \sum_1^m M_m\int_{l-b_m}^l\int_{l-b_m}^x dx\,dx\right)\right.$$
$$\left. + \left[V_A\int_0^l\int_0^x x\,dx\,dx + \sum_1^n P_n\int_{l-a_n}^l\int_{l-a_n}^x (x - l + a_n)\,dx\,dx\right]\right\}$$
$$= \frac{1}{6\,E\,I} M'''_{AB,\,B} \qquad (30)$$

in which M'''_{AB} is the third moment of the force system acting on the segment AB, including end moments and end shears of the segment, with respect to point B, the latter point (B) being on the right side of point A. The more general form may be written as

$$y_{BA} = \pm \frac{M'''_{AB,\,B}}{6\,E\,I} \qquad (31a)$$

which takes the positive sign when point B is on the right side of point A.

Referring again to Fig. 8, the deflection of point B is:

$$\delta_B = \delta_A + \theta_A\,l + y_{BA} = \delta_A + \theta_A\,l \pm \frac{M'''_{AB,\,B}}{6\,E\,I} \qquad (31b)$$

In Eq. 31b the positive sign is used when point B is on the right side of point A.

Eqs. 31 give the second fundamental theorem of the method of second and third moments, for beam deflections:

> Theorem VII. The deflection of any point B, measured from the tangent to the elastic curve at point A, of a segment (AB) of a straight beam of constant flexural rigidity, is equal to the third moment of the force system

acting on the segment AB, including end moments and end shears of the segment, taken with respect to point B, divided by six times the flexural rigidity; that is, by 6 $E\,I$.

Theorems VI and VII are similar in nature to theorems I and II of the moment-area method, and lead to identical results. The following examples illustrate the application of theorems VI and VII.

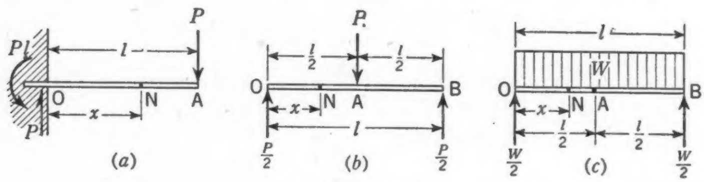


FIG. 10

Example 1. Cantilever Beam Loaded at Free End.—For the case shown in Fig. 10(*a*), $\theta_O = 0$ and $\delta_O = 0$. From theorem VI or Eq. 29*b*,

$$\theta_N = \frac{1}{2\,E\,I}\,M''_{O,N} = \frac{1}{2\,E\,I}\,(P\,x^2 - 2\,P\,l\,x) \quad \text{(32a)}$$

and, from theorem VII or Eq. 31*b*,

$$\delta_N = \frac{1}{6\,E\,I}\,M'''_{ON,N} = \frac{1}{6\,E\,I}\,(P\,x^3 - 3\,P\,l\,x^2) \quad \text{(32b)}$$

At the free end, A (see Fig. 10(*a*)):

$$\theta_A = -\frac{P\,l^2}{2\,E\,I};\ \text{and}\ \delta_A = -\frac{P\,l^3}{3\,E\,I} \quad \text{(33)}$$

Example 2. Simple Beam with a Concentrated Load at Center.—For the case shown in Fig. 10(*b*), as in Fig. 10(*a*), $\theta_A = 0$ and $\delta_O = 0$. Again, from theorem VI or Eq. 29*b*,

$$\theta_A = \theta_O + \frac{1}{2\,E\,I}\,M''_{OA} = \theta_O + \frac{1}{2\,E\,I}\left(\frac{P}{2}\right)\left(\frac{l}{2}\right)^2 = 0 \quad \text{(34a)}$$

and

$$\theta_O = -\frac{P\,l^2}{16\,E\,I} \quad \text{(34b)}$$

Thus, for $x \leqq l/2$, by Eqs. 29*b* and 31*b*:

$$\theta_N = \theta_O + \frac{1}{2\,E\,I}\,M''_{ON} = \frac{P}{16\,E\,I}\,(4\,x^2 - l^2) \quad \text{(35a)}$$

and

$$\delta_N = \theta_O\,x + \frac{1}{6\,E\,I}\,M'''_{ON,N} = \frac{1}{48\,E\,I}\,P\,x\,(4\,x^2 - 3\,l^2) \quad \text{(35b)}$$

At the center A, $x = \frac{1}{2}\,l$:

$$\delta_A = -\frac{1}{48\,E\,I}\,P\,l^3 \quad \text{(35c)}$$

Example 3. Simple Beam Subject to Uniformly Distributed Load.—By Eq. 29*b* with $\theta_A = 0$ and $\delta_O = 0$ (see Fig. 10(*c*)):

$$\theta_A = \theta_O + \frac{1}{2\,E\,I}\,M''_{OA} = \theta_O + \frac{1}{2\,E\,I}\left[\frac{W}{2}\left(\frac{l}{2}\right)^2 - \int_0^{l/2}\left(\frac{W}{l}\right)x^2\,dx\right]$$

$$= \theta_O + \frac{W\,l^2}{24\,E\,I} = 0 \ldots\ldots (36a)$$

and

$$\theta_O = -\frac{W\,l^2}{24\,E\,I} \ldots\ldots\ldots\ldots (36b)$$

By Eqs. 29*b* and 31*b*:

$$\theta_N = \theta_O + \frac{1}{2\,E\,I}\,M''_{ON} = -\frac{W\,l^2}{24\,E\,I} + \frac{1}{2\,E\,I}\left[\frac{W}{2}\,x^2 - \int_0^{x}\left(\frac{W}{l}\right)x^2\,dx\right]$$

$$= \frac{W}{24\,E\,I}\left(-\,l^2 + 6\,x^2 - 4\,\frac{x^3}{l}\right) \ldots\ldots (37a)$$

and

$$\delta_N = \delta_O + \theta_O\,x + \frac{1}{6\,E\,I}\,M'''_{ON,N} = -\frac{1}{24\,E\,I}\,W\,l^2\,x$$

$$+ \frac{1}{6\,E\,I}\left(\frac{W}{2}\,x^3 - \int_0^{x}\frac{W}{l}\,x^3\,dx\right) = \frac{1}{24\,E\,I}\,W\left(-\,l^2\,x + 2\,x^3 - \frac{x^4}{l}\right) \ldots (37b)$$

For $x = l/2$:

$$\delta_A = -\frac{5\,W\,l^3}{384\,E\,I} \ldots\ldots\ldots\ldots (38)$$

The computation of Eqs. 36 to 38 may be facilitated by using Eqs. 19, substituting $\frac{W}{l}\,x$ for W, and x for l. The second and third moments, referred to point N as moment center, due to uniform load only, are, respectively:

$$M'' = -\frac{W\,x^3}{3\,l}\,;\ \text{and}\ M''' = -\frac{W\,x^4}{4\,l} \ldots\ldots\ldots\ldots (39)$$

From which:

$$M''_{ON} = M''_{ON,N} = \frac{W}{2}\,x^2 - \frac{W\,x^3}{3\,l} \ldots\ldots (40a)$$

and

$$M'''_{ON,N} = \frac{W}{2}\,x^3 - \frac{W\,x^4}{4\,l} \ldots\ldots (40b)$$

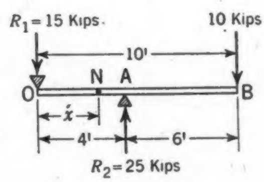


FIG. 11

Example 4. Overhanging Beam Loaded at Free End.—By Eq. 31*b*, with $\delta_O = 0$ and $\delta_A = 0$ (see Fig. 11):

$$\delta_A = \delta_O + \theta_O \times 4 + \frac{1}{6\,E\,I}\,M'''_{OA,A} = \theta_O \times 4 + \frac{1}{6\,E\,I}(-\,15 \times 4^3) = 0 \ldots (41a)$$

and

$$\theta_O = \frac{40}{E\,I} \qquad (41b)$$

For $x \leqq 4$ ft, Eqs. 29*b* and 31*b* yield:

$$\theta_N = \theta_O + \frac{1}{2\,E\,I}\,M''_{ON} = \frac{40}{E\,I} + \frac{1}{2\,E\,I}\,(-\,15)\,x^2 = \frac{1}{E\,I}\,(40 - 7.5\,x^2) \qquad (42a)$$

and

$$\delta_N = \delta_O + \theta_O\,x + \frac{1}{6\,E\,I}\,M'''_{ON,N} = \frac{1}{E\,I}\,(40\,x - 2.5\,x^3) \qquad (42b)$$

Similarly, for $x \geqq 4$ ft:

$$\theta_N = \theta_O + \frac{1}{2\,E\,I}\,M''_{ON} = \frac{1}{E\,I}\,(240 - 100\,x + 5\,x^2) \qquad (43a)$$

and

$$\delta_N = \delta_O + \theta_O\,x + \frac{1}{6\,E\,I}\,M'''_{ON,N}$$

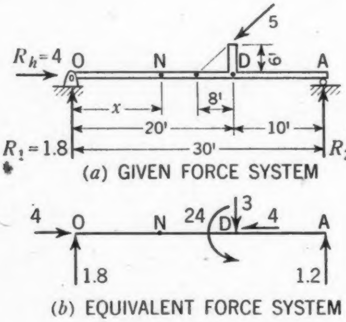
$$= \frac{1}{6\,E\,I}\,(10\,x^3 - 300\,x^2 + 1{,}440\,x - 1{,}600) \qquad (43b)$$

At the free end:

$$\theta_B = -\,\frac{260}{E\,I}\,;\ \text{and}\ \delta_B = -\,\frac{1{,}200}{E\,I} \qquad (44)$$

in which E is in kips per square foot, and I is in inch-feet[4].

Example 5. Simple Beam Subject to Indirect Loading.—The external load on the beam in Fig. 12(*a*) is replaced by the equivalent force system as shown



(*a*) GIVEN FORCE SYSTEM

(*b*) EQUIVALENT FORCE SYSTEM

FIG. 12

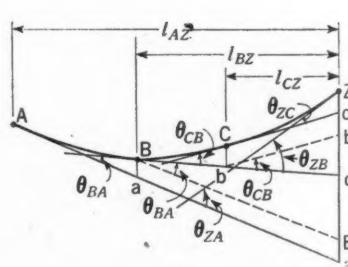


FIG. 13

in Fig. 12(*b*). By Eq. 31*b*, with $\delta_O = 0$ and $\delta_A = 0$:

$$\delta_A = \delta_O + \theta_O \times 30 + \frac{1}{6\,E\,I}\,M'''_{OA,A}$$

$$= 30\,\theta_O + \frac{1}{6\,E\,I}\,(1.8 \times 30^3 - 3 \times 10^3 - 3 \times 24 \times 10^2) = 0 \qquad (45a)$$

and

$$\theta_O = -\,\frac{640}{3\,E\,I} \qquad (45b)$$

for $x \leqq 20$ ft, by Eqs. 29*b* and 31*b*:

$$\theta_N = \theta_O + \frac{1}{2\,E\,I} M''_{ON} = -\frac{1{,}280}{6\,E\,I} + \frac{1}{2\,E\,I} \times 1.8\,x^2$$

$$= \frac{1}{6\,E\,I}(-\,1{,}280 + 5.4\,x^2) \qquad \text{(46a)}$$

and

$$\delta_N = \delta_O + \theta_O\,x + \frac{1}{6\,E\,I} M'''_{ON,\,N} = \frac{x}{6\,E\,I}(-\,1{,}280 + 1.8\,x^2) \qquad \text{(46b)}$$

For $x \geqq 20$ ft:

$$\theta_N = \theta_O + \frac{M''_{ON}}{2\,E\,I} = -\frac{1}{E\,I}(0.6\,x^2 - 36\,x + 333.3) \qquad \text{(47a)}$$

and

$$\delta_N = \delta_O + \theta_O\,x + \frac{1}{6\,E\,I} M'''_{ON,\,N}$$

$$= -\frac{1}{6\,E\,I}(1.2\,x^3 - 108\,x^2 + 2{,}000\,x + 4{,}800) \qquad \text{(47b)}$$

Straight Beams of Segmentally Constant Flexural Rigidity

Fig. 13 shows the elastic curve of a beam whose flexural rigidity is $(E\,I)_{AB}$ for segment AB, $(E\,I)_{BC}$ for segment BC, and $(E\,I)_{CZ}$ for segment CZ. By Eq. 29*a*:

$$\theta_{ZA} = \theta_{BA} + \theta_{CB} + \theta_{ZC} = \frac{1}{2(E\,I)_{AB}} M''_{AB}$$

$$+ \frac{1}{2(E\,I)_{BC}} M''_{BC} + \frac{1}{2(E\,I)_{CZ}} M''_{CZ} \qquad \text{(48a)}$$

The general form, therefore, is:

$$\theta_{ZA} = \sum \frac{M''_{AB}}{2(E\,I)_{AB}} \qquad \text{(48b)}$$

Substituting, $M''_{BC} = M''_{AC} - M''_{AB}$; and $M''_{CZ} = M''_{AZ} - M''_{AC}$. In Eq. 48*b*:

$$\theta_{ZA} = \frac{1}{2(E\,I)_{AB}} M''_{AB} + \frac{1}{2(E\,I)_{BC}}(M''_{AC} - M''_{AB})$$

$$+ \frac{1}{2(E\,I)_{CZ}}(M''_{AZ} - M''_{AC}) = \left[\frac{1}{2(E\,I)_{AB}} - \frac{1}{2(E\,I)_{BC}}\right] M''_{AB}$$

$$+ \left[\frac{1}{2(E\,I)_{BC}} - \frac{1}{2(E\,I)_{CZ}}\right] M''_{AC} + \frac{1}{2(E\,I)_{CZ}} M''_{AZ} = \Delta\left(\frac{1}{2\,E\,I}\right)_B M''_{AB}$$

$$+ \Delta\left(\frac{1}{2\,E\,I}\right)_C M''_{AC} + \Delta\left(\frac{1}{2\,E\,I}\right)_Z M''_{AZ} = \sum \Delta\left(\frac{1}{2\,E\,I}\right)_B M''_{AB} \qquad \text{(48c)}$$

in which

$$\Delta\left(\frac{1}{2\,E\,I}\right)_B = \frac{1}{2(E\,I)_{AB}} - \frac{1}{2(E\,I)_{BC}} \qquad \text{(49a)}$$

$$\Delta\left(\frac{1}{2\,E\,I}\right)_C = \frac{1}{2(E\,I)_{BC}} - \frac{1}{2(E\,I)_{CZ}} \qquad \text{(49b)}$$

and

$$\Delta\left(\frac{1}{2\,E\,I}\right)_Z = \frac{1}{2(E\,I)_{CZ}} - 0 = \frac{1}{2(E\,I)_{CZ}} \qquad (49c)$$

Eq. 48*b* or Eq. 48*c* may be used, but the latter is the simpler.

Referring to Fig. 13, Eqs. 29*a* and 31*a*:

$$y_{ZA} = \overline{a'Z} = \overline{a'B'} + \overline{B'd} + \overline{db'} + \overline{b'c'} + \overline{c'Z} = y_{BA} + \theta_{BA}\, l_{BZ}$$
$$+ \theta_{CB}\, l_{CZ} + y_{CB} + y_{ZC} = \frac{1}{6(E\,I)_{AB}} M'''_{AB,B} + \frac{1}{2(E\,I)_{AB}} M''_{AB}\, l_{BZ}$$
$$+ \frac{1}{2(E\,I)_{BC}} M''_{BC}\, l_{CZ} + \frac{1}{6(E\,I)_{BC}} M'''_{BC,C} + \frac{1}{6(E\,I)_{CZ}} M'''_{CZ,Z}$$
$$= \frac{1}{6(E\,I)_{AB}} (M'''_{AB,B} + 3\,M''_{AB}\, l_{BZ}) + \frac{1}{6(E\,I)_{BC}} (M'''_{BC,C} + 3\,M''_{BC}\, l_{CZ})$$
$$+ \frac{1}{6(E\,I)_{CZ}} M'''_{CZ,Z} \qquad (50)$$

By the second characteristic of second and third moments, Eq. 12 yields:

$$M'''_{AB,Z} = M'''_{AB,B} + 3\,M''_{AB}\, l_{BZ} \qquad (51a)$$

and

$$M'''_{BC,Z} = M'''_{BC,C} + 3\,M''_{BC}\, l_{CZ} \qquad (51b)$$

Substituting Eqs. 51 in Eq. 50:

$$y_{ZA} = \frac{1}{6(E\,I)_{AB}} M'''_{AB,Z} + \frac{1}{6(E\,I)_{BC}} M'''_{BC,Z} + \frac{1}{6(E\,I)_{CZ}} M'''_{CZ,Z} \qquad (52a)$$

Therefore, the general formula is:

$$y_{ZA} = \sum \frac{1}{6(E\,I)_{AB}} M'''_{AB,Z} \qquad (52b)$$

Substituting—

$$M'''_{BC,Z} = M'''_{AC,Z} - M'''_{AB,Z} \qquad (53a)$$

and

$$M'''_{CZ,Z} = M'''_{AZ,Z} - M'''_{AC,Z} \qquad (53b)$$

—in Eq. 52*b*:

$$y_{ZA} = \frac{1}{6(E\,I)_{AB}} M'''_{AB,Z} + \frac{1}{6(E\,I)_{BC}} (M'''_{AC,Z} - M'''_{AB,Z})$$
$$+ \frac{1}{6(E\,I)_{CZ}} (M'''_{AZ,Z} - M'''_{AC,Z}) \approx \left[\frac{1}{6(E\,I)_{AB}}\right.$$
$$\left. - \frac{1}{6(E\,I)_{BC}}\right] M'''_{AB,Z} + \left[\frac{1}{6(E\,I)_{BC}} - \frac{1}{6(E\,I)_{CZ}}\right] M'''_{AC,Z}$$
$$+ \frac{1}{6(E\,I)_{CZ}} M'''_{AZ,Z} = \Delta\left(\frac{1}{6\,E\,I}\right)_B M'''_{AB,Z}$$
$$+ \Delta\left(\frac{1}{6\,E\,I}\right)_C M'''_{AC,Z} + \Delta\left(\frac{1}{6\,E\,I}\right)_Z M'''_{AZ,Z}$$
$$= \sum \Delta\left(\frac{1}{6\,E\,I}\right)_B M'''_{AB,Z} \qquad (54)$$

in which (compared with Eqs. 49):

$$\Delta\left(\frac{1}{6\,E\,I}\right)_B = \frac{1}{6(E\,I)_{AB}} - \frac{1}{6(E\,I)_{BC}} \quad \text{(55a)}$$

$$\Delta\left(\frac{1}{6\,E\,I}\right)_C = \frac{1}{6(E\,I)_{BC}} - \frac{1}{6(E\,I)_{CZ}} \quad \text{(55b)}$$

and

$$\Delta\left(\frac{1}{6\,E\,I}\right)_Z = \frac{1}{6(E\,I)_{CZ}} \quad \text{(55c)}$$

For beams of variable flexural rigidity, Eqs. 48c and 54 become:

$$\theta_{ZA} = \int_{x=0}^{x=l} M''_{Ax}\, d\left[\frac{1}{2(E\,I)_x}\right] \quad \text{(56a)}$$

and

$$y_{ZA} = \int_{x=0}^{x=l} M'''_{Ax,\,x}\, d\left[\frac{1}{6(E\,I)_x}\right] \quad \text{(56b)}$$

in which $(E\,I)_x$, M''_{Ax}, and $M'''_{Ax,\,x}$ are functions of x. Solutions of problems by Eqs. 56 are naturally somewhat complicated. The following examples illustrate the application of Eqs. 48, 52, and 55.

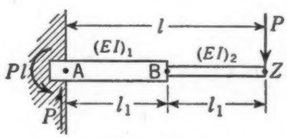


FIG. 14

Example 6. Cantilever Beam with Segmentally Constant Flexural Rigidity.—By Eqs. 48c and 54 (see Fig. 14):

$$\theta_Z = \Delta\left(\frac{1}{2\,E\,I}\right)_B M''_{AB} + \frac{1}{2(E\,I)_2} M''_{AZ} = \left[\frac{1}{2(E\,I)_1} - \frac{1}{2(E\,I)_2}\right](P\,l^2_1 - 2\,P\,l \times l_1) + \frac{1}{2(E\,I)_2}(P\,l^2 - 2\,P\,l \times l) = -\frac{P\,l^3}{8}\left[\frac{3}{(E\,I)_1} - \frac{1}{(E\,I)_2}\right] \quad \text{(57a)}$$

and

$$\delta_Z = \Delta\left(\frac{1}{6\,E\,I}\right)_B M'''_{AB,\,Z} + \frac{1}{6(E\,I)_2} M'''_{AZ,\,Z} = \left[\frac{1}{6(E\,I)_1} - \frac{1}{6(E\,I)_2}\right] \times [P\,l^3_1 - 3\,P\,l \times l^2_1 + 3\,(P\,l^2_1 - 2\,P\,l \times l_1)\,l_1] + \frac{1}{6(E\,I)_2}(P\,l^3 - 3\,P\,l \times l^2) = -\frac{P\,l^3}{24}\left[\frac{7}{(E\,I)_1} + \frac{1}{(E\,I)_2}\right] \quad \text{(57b)}$$

Example 7. Simply Supported Beam with Segmentally Constant Flexural Rigidity and a Load at the Center.—Referring to Fig. 15, Eq. 52b yields:

$$\delta_A = y_{ZA} = \frac{1}{6(E\,I)_1} M'''_{AB,\,Z} + \frac{1}{6(E\,I)_0} M'''_{BZ,\,Z} = \frac{P\,l^3}{384}\left[\frac{7}{(E\,I)_1} + \frac{1}{(E\,I)_0}\right] \quad \text{(58)}$$

Statically Indeterminate Beams and Frames

The following examples illustrate the use of the method of second and third moments to analyze statically indeterminate structures.

Example 8. Beam Fixed at One End and Simply Supported at the Other End, With a Concentrated Load at the Center.—Referring to Fig. 16(a), with $\theta_O = 0$

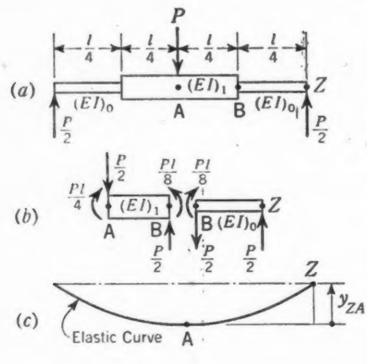


FIG. 15

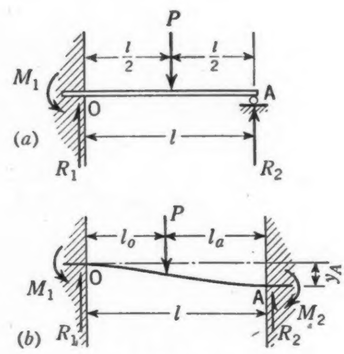


FIG. 16

and $\delta_A = 0$, Eq. 31a yields:

$$\delta_A = y_{AO} = \frac{M'''_{OA,A}}{6\,E\,I} = 0 \ldots\ldots\ldots\ldots\ldots\ldots \text{(59)}$$

According to Eq. 12:

$$M'''_{OA,A} = M'''_{OA,O} + 3\,M''_{OA}\,l \ldots\ldots\ldots\ldots\ldots \text{(60a)}$$

and Eq. 10b,

$$M''_{OA} = M''_{OA,O} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(60b)}$$

Combining Eqs. 60:

$$M'''_{OA,A} = M'''_{OA,O} + 3\,M''_{OA,O}\,l = P\left(\frac{l}{2}\right)^3 - R_2\,l^3 + 3\left[-P\left(\frac{l}{2}\right)^2 + R_2\,l^2\right] l = 0 \ldots \text{(61)}$$

From which:

$$R_2 = \tfrac{5}{16}\,P; \qquad R_1 = \tfrac{11}{16}\,P; \text{ and } M_1 = \tfrac{3}{16}\,P\,l \ldots\ldots\ldots \text{(62)}$$

Example 9. Beam Fixed at Both Ends But Displayed Vertically through a Distance y, with a Concentrated Load at Center.—Referring to Fig. 16(b), with $\theta_{AO} = 0$ and $M''_{OA,A} = 0$:

$$R_1\,l^2 - 2\,M_1\,l - P\,l^2_a = 0 \ldots\ldots\ldots\ldots\ldots\ldots \text{(63a)}$$

Since $\delta_{AO} = -\,y_A$ and $M'''_{OA,A} = -\,6\,E\,I\,y_A$:

$$R_1\,l^3 - 3\,M_1\,l^2 - P\,l^3_a = -\,6\,E\,I\,y_A \ldots\ldots\ldots\ldots \text{(63b)}$$

Combining Eqs. 62:

$$M_1 = \frac{1}{l^2}\left[P\, l_a^2\,(l - l_a) + 6\, E\, I\, y_A\right] \ldots\ldots\ldots\ldots\ldots (64a)$$

and

$$R_1 = \frac{1}{l^3}\left(3\, P\, l_a^2\, l - 2\, P\, l_a^3 + 12\, E\, I\, y_A\right) \ldots\ldots\ldots\ldots (64b)$$

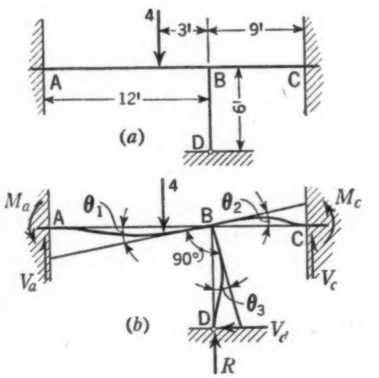


FIG. 17

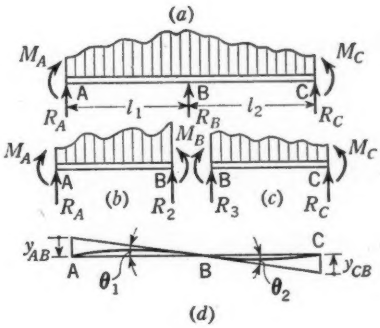


FIG. 18

Example 10. Continuous Frame From the Static Condition of Equilibrium (See Fig. 17).—With $M_B = 0$:

$$12\, V_a - M_a - 12 + 6\, V_d - M_c + 9\, V_c = 0 \ldots\ldots\ldots\ldots (65)$$

From Eqs. 29a and 31a:

$$\theta_1 = \frac{1}{2\, E\, I}\, M''_{AB,\,B} = \frac{1}{2\, E\, I}\,(V_a \times 12^2 - 2\, M_a \times 12 - 4 \times 3^2)$$

$$= \frac{1}{2\, E\, I}\,(144\, V_a - 24\, M_a - 36) \ldots . (66a)$$

$$\theta_2 = \frac{1}{2\, E\, I}\, M''_{BC,\,B} = \frac{1}{2\, E\, I}\,(18\, M_c - 81\, V_c) \ldots\ldots\ldots\ldots (66b)$$

and

$$\theta_3 = \frac{y_{DB}}{6} = \frac{1}{36\, E\, I}\, M'''_{BD,\,D} = \frac{1}{36\, E\, I}\,(V_d \times 6^3 - 3 \times V_d$$

$$\times\, 6^2 \times 6) = -\frac{1}{2\, E\, I}\, 24\, V_d \ldots . (66c)$$

When $\theta_1 = -\,\theta_2$, equating Eqs. 66a and 66b yields $144\, V_a - 24\, M_a - 36 = -\,18\, M_c + 81\, V_c$, which reduces to:

$$8\, M_a - 48\, V_a - 6\, M_c + 27\, V_c = -\,12 \ldots\ldots\ldots\ldots (67a)$$

When $\theta_2 = \theta_3$, equating Eqs. 66b and 66c yields $18\, M_c - 81\, V_c = -\,24\, V_d$, which reduces to:

$$6\, M_c - 27\, V_c + 8\, V_d = 0 \ldots\ldots\ldots\ldots\ldots\ldots (67b)$$

Therefore, since $y_{BA} = \frac{1}{6\,E\,I}\,M'''_{AB,B} = 0$; and $M'''_{AB,B} = V_a \times 12^3 - 3M_a \times 12^2 - 4 \times 3^3 = 0$; or

$$V_a = \tfrac{1}{16} + \frac{M_a}{4} \quad \text{(68a)}$$

Similarly, $y_{BC} = \frac{1}{6\,E\,I}\,M'''_{BC,B} = 0$; and $M'''_{BC,B} = V_c \times 9^3 - 3\,M_c \times 9^2 = 0$.

Therefore,

$$V_c = \frac{M_c}{3} \quad \text{(68b)}$$

Substituting Eqs. 68 into Eq. 65:

$$8\,M_a + V_d + 8\,M_c = 45 \quad \text{(69a)}$$

substituting Eqs. 68 into Eq. 67*a*:

$$4\,M_a - 3\,M_c = 9 \quad \text{(69b)}$$

and, substituting Eq. 68*b* into Eq. 67*b*:

$$8\,V_d - 3\,M_c = 0 \quad \text{(69c)}$$

Solving Eqs. 69 simultaneously: $M_c = 1.174$ kip-ft; $M_a = 3.13$ kip-ft; and $V_d = 0.44$ kips. From Eqs. 68 and from static conditions of equilibrium: $V_c = 0.391$ kips; $V_a = 0.844$ kips; and $R = 2.766$ kips.

Example 11. Derivation of the Three-Moment Equation.—Fig. 18 shows a continuous beam of two spans subject to any transverse load. For simplicity, assume that the supports A, B, and C are unyielding, so that $\theta_1 = -\theta_2$. By geometry, $\frac{y_{AB}}{l_1} = -\frac{y_{CB}}{l_2}$; or $\frac{-M'''_{AB,A}}{6\,E\,I\,l_1} = -\frac{+M'''_{BC,C}}{6\,E\,I\,l_2}$. Simplifying:

$$\frac{1}{l_1}\,M'''_{AB,A} = \frac{1}{l_2}\,M'''_{BC,C} \quad \text{(70a)}$$

Eq. 70*a* may be written in the following form:

$$\frac{1}{l_1}(-R_2\,l^3_1 - 3\,M_B\,l^2_1 + M'''_{\overline{AB},A}) = \frac{1}{l_2}(R_3\,l^3_2 + 3\,M_B\,l^2_2 + M'''_{\overline{BC},C}) \quad \text{(70b)}$$

in which $M'''_{\overline{AB},A}$ represents the third moment, with respect to point A, of the external loads only, on the span AB, end shears and end moments being excluded; and $M'''_{\overline{BC},C}$ represents the third moment, with respect to point C, of the external loads only, on the span BC. Similarly, $M_{\overline{AB},A}$ and $M''_{\overline{AB},A}$ represent the first and second moments, respectively, with reference to point A, of the external loads only, on span AB. For span AB (Fig. 18(*b*)): $M_{AB,A} = M_A - M_B - R_2\,l_1 + M_{\overline{AB},A} = 0$; and

$$R_2 = \frac{1}{l_1}(M_A - M_B + M_{\overline{AB},A}) \quad \text{(71a)}$$

For span BC (Fig. 18(c)): $M_{BC,C} = M_B - M_C + R_3 l_2 + M_{\overline{CB},C} = 0$; and

$$R_3 = \frac{1}{l_2}(M_C - M_B - M_{\overline{BC},C}) \ldots\ldots\ldots\ldots\ldots\ldots (71b)$$

Substituting Eqs. 71 in Eq. 70*b*, the three-moment equation is derived, thus:

$$l_1 M_A + 2(l_1 + l_2) M_B + l_2 M_C = \left(\frac{M'''_{\overline{AB},A}}{l_1} - \frac{M'''_{\overline{BC},C}}{l_2}\right)$$
$$- (l_1 M_{AB,A} - l_2 M_{\overline{BC},C} \ldots (72)$$

Example 12. Continuous Beam.—The problem in Fig. 19 may be solved with the three-moment equation. By Eq. 72, $2(5 + 10) M_B = \left[\frac{500 \times 3^3}{5} - \frac{(-12{,}000) \times 10^3}{4 \times 10}\right] - [5 \times 500 \times 3 - 10\ (-12{,}000) \times 5] = -304{,}800$ lb-ft; and $M_B = -10{,}160$ lb-ft. For span AB (Fig. 19(c)): $M_{BC,B} = -R_3 \times 10 + 12{,}000 \times 5 - 10{,}160 = 0$. Consequently, $R_3 = 4{,}984$ lb; and $R_2 = 9{,}348$ lb.

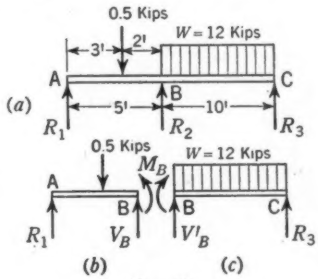


FIG. 19

ACKNOWLEDGMENTS

The writer is indebted to Prof. Fang-Yin Tsai, formerly dean of the College of Engineering, and Prof. T. M. Yu, formerly of the department of civil engineering, both of the National Chung Cheng University, Nanchang, Kiangsi, China, for their invaluable help in the preparation of the paper.

APPENDIX. NOTATION

The following letter symbols, used in the paper and in its discussions, conform essentially to American Standard Letter Symbols for Mechanics, Structural Engineering and Testing Materials (ASA—Z10a—1932), prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1932:

a = lever arm of a point force;
b = reference distance to the center of a couple;
E = modulus of elasticity;
I = moment of inertia;
l = span length, or (where stated) equivalent length of a uniform load;
M = moment:
  M'' = second moment;
  M''' = third moment;
  M_c = a couple, or torque;
m = a variable number;

n = a variable number;
P = point force or concentrated load, with appropriate subscripts to denote the point of application;
R = resultant force;
V = vertical shear, or total shear force;
W = total distributed weight, or load;
w = distributed load per unit distance;
x = variable distances parallel to the x-axis;
y = variable distances and deflections parallel to the y-axis;
z = variable distances along a beam, to define the location of an increment dz of distributed load;
δ = vertical deflections (see also y); and
θ = angular deflection.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

A RATIONAL EXPLANATION OF COLUMN BEHAVIOR

BY FREDERICK L. RYDER,[1] ESQ.

SYNOPSIS

The general method of approach is to analyze, mathematically, the elastic behavior of the simple ideal column, and then to apply the results, with suitable modifications, to the practical column. The basic differential equation is first set up, and then is integrated to yield the fundamental column formula relating axial force, moment of inertia, modulus of elasticity, length, and center deflection (Section 1).

For a given P/A-value, the variation of center deflection with the L/r-ratio is studied, for conditions above the critical point. The projected length corresponding to any value of center deflection is calculated, thus affording an approximate idea of the deflected column shape. A chart relates values of P/A, L/r, projected length, and center deflection. Finally, the exact shape of a typical ideal column is computed, and is shown to be similar for all ideal columns having the same relative center deflection (Section 2).

It is next shown that, above the critical point, the basic differential equation is satisfied if the column is (*a*) straight, or (*b*) deflected to the position dictated by the fundamental column formula. The stability of the column in these two positions, and in fact in any position, is analyzed by a detailed study of the internal moments caused by deflection and the external moments caused by the axial force. It is shown that only one position of the column is stable—the position corresponding to the fundamental column formula. The analysis also re-emphasizes the well-known fact that at the critical point the deflection is indeterminate (Section 3).

For conditions below the critical point, a chart relates the factor of safety of an ideal column to the P/A-value and the L/r-value. The differences between the practical and the ideal column are next discussed; and the analytical

NOTE.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by August 1, 1947.

[1] Development Engr., Shirgun Corp., New York, N. Y.

solution for the deflection and stresses of a column having end eccentricity, initial deflection, and transverse loading is obtained. The "step-by-step" method for the numerical solution of practical column problems is then presented in detail and is applied to two typical problems (Section 4). Stresses and factor of safety in a column loaded above the critical point are discussed (Section 5).

Introduction

Most of the familiar and widely used column formulas are essentially based on the results of tests. They are convenient for predicting column action under certain important conditions, but they are not intended to present the rationale of column behavior. For example, some questions that cannot be answered by ordinary column formulas are:

A. Why do columns buckle suddenly at the critical load?
B. What are the actual stresses in the column?
C. How do columns act above the critical load?

These questions, and others of similar interest, can be answered by an a priori mathematical analysis, without recourse to test results. In fact, a complete solution of the elastic behavior of a simple column can be deduced mathematically from the given statical condition of the column. It is the writer's purpose to trace the mathematical steps leading to this solution and to interpret the results in terms most useful to the average engineer.

As a supplement to the theoretical discussion, the powerful and universally adaptable step-by-step method of solving practical column problems is included in the argument on actual stress and factor of safety.

The scope of this paper, then, is twofold:

1. To explain the mechanism of column behavior by mathematical analysis; and

2. To present a widely applicable method for the solution of practical column problems.

Notation.—The letter symbols in this paper are defined where they first appear, in the text or by diagram, and are assembled alphabetically in the Appendix. Discussers are requested to adapt their notation to this form.

1. The Fundamental Differential Equation and Its Integration

The column shown in Fig. 1(*a*) is assumed to be perfectly straight initially, and is of homogeneous material having the modulus of elasticity, E. The cross section and therefore the moment of inertia, I, are constant. The resultant of the load P acts through the centroid of the cross section, thus insuring that P does not produce moment at the ends of the column. The length of the column, measured along the curve, equals L.

The column is shown in a transversely deflected position, δ being the deflection at the center. At any point, such as A, the column feels the moment:

$$M = P\,y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1a)$$

According to familiar theory in the strength of materials, the moment at any point of an initially straight member equals:

$$M = -E\,I \times (\text{curvature}) \ldots\ldots\ldots\ldots\ldots\ldots (1b)$$

For present purposes, curvature is advantageously considered the "rate of change of slope per unit of length along the curve." Referring to the triangle

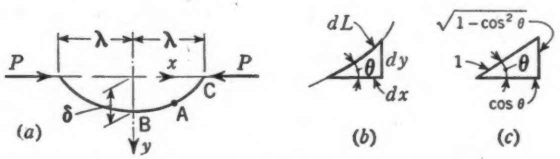


FIG. 1

of infinitesimals, Fig. 1(*b*):

$$(\text{Curvature}) = \frac{d\theta}{ds} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1c)$$

in which ds is the differential of length of the curve; and θ is the angle of slope or

$$\theta = \tan^{-1}\frac{dy}{dx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Combining Eqs. 1:

$$P\,y = -E\,I \times \frac{d\theta}{ds} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Eq. 3 is the fundamental differential equation of the elastic curve of the column, and forms the basis for the present analysis.

Certain assumptions and omissions are implicit in Eq. 1*b*:

a. Only the curvature caused by bending moment is considered in Eq. 1*b*. With the column in a deflected position, the curvature will be somewhat affected by shearing forces. However, the curvature caused by shear is usually very small compared to the curvature caused by moment, and will therefore be neglected.

b. Eq. 1*b* is not strictly true for a member with appreciable curvature.[2] The error is not important, however, if the ratio ρ/c remains larger than about 10, in which ρ is the radius of curvature and c is the distance from the neutral axis to the extreme fibers. For small deflections this error is negligible. Large deflections, and consequently large curvatures, will affect the results slightly.

c. The modulus of elasticity, E, is assumed to be constant, necessitating the maximum stress in the column to be less than the proportional limit of the material.

---

[2] "Advanced Mechanics of Materials," by Fred B. Seely, John Wiley & Sons, Inc., New York, N. Y., 1932, Chapter VII.

Eq. 1*b* is often used in the simplified form:

$$M = -E\,I\,\frac{d^2y}{dx^2} \dots\dots\dots\dots\dots\dots\dots\dots (4)$$

in which $\frac{d^2y}{dx^2}$ is made equal to the curvature. The true expression for the curvature in terms of $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ (see any calculus text) is:

$$(\text{Curvature}) = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}} \dots\dots\dots\dots\dots\dots (5)$$

The curvature is $\frac{d^2y}{dx^2}$ only when $\frac{dy}{dx} = 0$—that is, when the member is perfectly straight. Therefore, Eq. 4 is not valid when the member has an appreciable deflection. The simplified Eq. 4 yields limited and imperfect results in the case of an ideal column, but is basically important in the solution of problems involving the practical column.

The first step in the integration of Eq. 3 is the elimination of one of its three variables. By Fig. 1(*b*),

$$ds = \frac{dy}{\sin\theta} \dots\dots\dots\dots\dots\dots\dots\dots (6a)$$

Substitution in Eq. 3 yields:

$$P\,y = -E\,I\,\frac{d\theta}{dy}\sin\theta \dots\dots\dots\dots\dots\dots\dots (6b)$$

In Eq. 6*b*, if—

$$k^2 \equiv \frac{P}{E\,I} \dots\dots\dots\dots\dots\dots\dots\dots (6c)$$

—it follows that

$$k^2\,y\,dy = -\sin\theta\,d\theta \dots\dots\dots\dots\dots\dots\dots (6d)$$

Integrating Eq. 6*d*,

$$\frac{k^2\,y^2}{2} + C = \cos\theta \dots\dots\dots\dots\dots\dots\dots (7)$$

in which C is an arbitrary constant. If $y = \delta$ then $\theta = 0$ (by symmetry), and $\cos\theta = 1$. Substituting these values in Eq. 7:

$$C = 1 - \frac{k^2\,\delta^2}{2} \dots\dots\dots\dots\dots\dots\dots (8a)$$

Rewriting Eq. 7:

$$\cos\theta = 1 - \frac{k^2}{2}(\delta^2 - y^2) \dots\dots\dots\dots\dots\dots (8b)$$

Eq. 8*b*, the result of the first integration, expresses a relation between the angle of slope (θ), the deflection at the center (δ), and the deflection (y) at

any point. The object of the next integration is to eliminate θ, an unknown, and to introduce L, the total length of the deflected column, into the equations. By similarity (Figs. 1(*b*) and 1(*c*)):

$$\frac{ds}{1} = \frac{dy}{\sqrt{1 - \cos^2\theta}} = \frac{dy}{\sqrt{(1 + \cos\theta)(1 - \cos\theta)}} \qquad (9a)$$

Using Eq. 8*b*:

$$ds = \frac{dy}{\sqrt{\left[2 - \frac{k^2}{2}(\delta^2 - y^2)\right]\left[\frac{k^2}{2}(\delta^2 - y^2)\right]}} \qquad (9b)$$

Eq. 9*b* may be simplified by writing:

$$y \equiv \delta\cos\phi \qquad (10a)$$

$$dy = -\delta\sin\phi\, d\phi \qquad (10b)$$

and

$$\delta^2 - y^2 = \delta^2 - \delta^2\cos^2\phi = \delta^2\sin^2\phi \qquad (10c)$$

Then

$$ds = \frac{-\delta\sin\phi\, d\phi}{\sqrt{\left(2 - \frac{k^2\,\delta^2}{2}\sin^2\phi\right)\left(\frac{k^2\,\delta^2}{2}\sin^2\phi\right)}} = \frac{d\phi}{k\sqrt{1 - \frac{k^2\,\delta^2}{4}\sin^2\phi}} \qquad (11)$$

The minus sign disappears in the numerator of Eq. 11 because $\sqrt{\delta^2}$ is selected as $-\delta$. Writing:

$$q \equiv \frac{k\,\delta}{2} \qquad (12a)$$

it follows that

$$k\, ds = \frac{d\phi}{\sqrt{1 - q^2\sin^2\phi}} \qquad (12b)$$

The integral of the right-hand term of Eq. 12*b* is known as an elliptic integral.[3] It can be evaluated only between definite limits. Points B and C in Fig. 1(*a*) are selected as the lower and upper limits, respectively.

Applying Eq. 10*a* at point B where $y = \delta$, it is found that $\cos\phi = \frac{y}{\delta} = \frac{\delta}{\delta} = 1$, from which $\phi = 0$. Likewise, at point C, $y = 0$, $\cos\phi = \frac{0}{\delta} = 0$, and $\phi = \frac{\pi}{2}$. The arc lengths s are 0 and $L/2$ at points B and C, respectively. With these quantities in Eq. 12*b*:

$$k\int_0^{L/2} ds = \frac{k\,L}{2} = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - q^2\sin^2\phi}} \equiv K \qquad (13)$$

Tabular values of K for different values of q have been prepared by B. O. Peirce.[4]

[3] "Advanced Calculus," by E. B. Wilson, Ginn & Co., Boston, Mass., 1912, p. 507.

[4] "A Short Table of Integrals," by B. O. Peirce, Ginn & Co., Boston, Mass., 3d Ed., 1929, p. 121.

Eq. 13, the fundamental column formula, expresses a relation between q, k, and L. For example, if $k^2 \equiv \frac{P}{E\,I}$ and L are known, then it becomes possible to solve for $q \equiv \frac{k\,\delta}{2}$, and hence δ can be ascertained. The implications of Eq. 13, with regard to practical engineering problems, are treated in Section 2.

2. Column Deflections Under Various Conditions

A pressing question confronts the engineer in regard to the behavior of the column: "What is the relationship of the deflection to the load P and to the physical constants E, I, and L?" With the aid of the fundamental column formula, Eq. 13, it is possible to answer this question by plotting δ/L, the relative deflection at the center, against $L/\hat{r}$, the ratio of the length to the minimum radius of gyration of the cross section. The ratio $L/\hat{r}$, of course, is the accepted criterion of stiffness for a column. (Incidentally, $\hat{r}$ should be regarded as a measure of the efficacy of distribution of the elements of cross-sectional area of the column, in so far as resistance to buckling is concerned.)

TABLE 1.—Center Deflection and Projected Length of a Simple Column

$\sin^{-1} q$ (degrees)	K [a]	q	$L/\hat{r} = 245\,K$	$\delta/L = q/K$	J [a]	$\lambda/L = \frac{J}{K} - \frac{1}{2}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)
0	1.57	0	384	0	1.57	0.50
30	1.69	0.5	415	0.296	1.47	0.37
60	2.16	0.866	529	0.401	1.21	0.06
75	2.77	0.966	676	0.35	1.08	−0.11
89	5.43	1.0	1,330	0.185	1.00	−0.316
90	∞	1.0	∞	0	1.00	−0.50

[a] "A Short Table of Integrals," by B. O. Peirce, Ginn & Co., Boston, Mass., 3d Ed., 1929, p. 121.

To introduce $\hat{r}$ into Eq. 6*c*, substitute $I = A\,\hat{r}^2$ and solve for k:

$$k = \frac{\sqrt{\frac{P}{A}}}{\hat{r}\,\sqrt{E}} \quad \text{(14)}$$

in which P/A is familiar to engineers as the direct compressive unit stress caused by P when the column is straight.

Consider a steel column with $E = 30{,}000{,}000$ lb per sq in. and $P/A = 2{,}000$ lb per sq in. Then, by Eq. 14:

$$k = \frac{\sqrt{2{,}000}}{\hat{r}\,\sqrt{30{,}000{,}000}} = \frac{1}{122.5\,\hat{r}} \quad \text{(15a)}$$

Referring to Eq. 13: $K = \frac{k\,L}{2} = \frac{L}{2 \times 122.5\,\hat{r}}$, or

$$\frac{L}{\hat{r}} = 245\,K \quad \text{(15b)}$$

By solving Eqs. 12*a* and 13 to eliminate k:

$$\frac{\delta}{L} = \frac{q}{K} \quad \text{(16)}$$

Eq. 16 makes it possible to compile Cols. 1 to 5 of Table 1. The value of K corresponding to $\sin^{-1} q = 90°$ does not appear in Mr. Peirce's table;[4] but it can be derived easily by using Eq. 13, thus: $K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - 1 \sin^2 \phi}} = \int_0^{\pi/2} \sec \phi \, d\phi$ $= \frac{1}{2}\left[\left(\log \frac{1 + \sin \phi}{1 - \sin \phi}\right)\right]_0^{\pi/2} = \frac{1}{2} \log \frac{\frac{1+1}{1+0}}{\frac{1-1}{1-0}} = \infty.$ The values of δ/L are plotted against those of L/r in Fig. 2. To assist in the interpretation of this curve, some idea should be formed as to the actual shape of the column corresponding to various plotted points. This is possible by solving for the different values of λ/L, in which λ is half the projection of the column on the x-axis (see Fig. 1(*a*)). Having found λ/L, and knowing δ/L from the curve of Fig. 2, the shape of the column, between the end points and the center, can be sketched. It is possible to derive the exact shape of the entire column (see Section 2 under "Exact Shape of the Deflected Column"), but this is not necessary for present purposes.

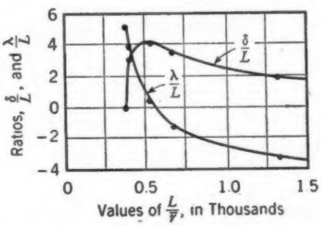


FIG. 2

Referring to Eq. 8*b*, and noting the similarity between the triangles of Figs. 1(*b*) and 1(*c*), the quantity x is introduced into the equations by writing:

$$\frac{dx}{ds} = \cos \theta = 1 - \frac{k^2}{2} (\delta^2 - y^2) \quad \text{(17a)}$$

To eliminate s (see Fig. 1(*b*)):

$$ds = \sqrt{(dx)^2 + (dy)^2} = dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad \text{(17b)}$$

Substituting Eq. 17*b* in Eq. 17*a*: $\frac{dx}{dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = \cos \theta;$

$$\frac{dy}{dx} = \sqrt{\frac{1}{\cos^2 \theta} - 1} = \sqrt{\frac{1 - \cos^2 \theta}{\cos^2 \theta}}; \qquad \text{or} \qquad \frac{dy \cos \theta}{\sqrt{(1 + \cos \theta)(1 - \cos \theta)}} = dx.$$

In other words,

$$dx = \frac{\left[1 - \frac{k^2}{2} (\delta^2 - y^2)\right] dy}{\sqrt{\left[2 - \frac{k^2}{2} (\delta^2 - y^2)\right]\left[\frac{k^2}{2} (\delta^2 - y^2)\right]}} \quad \text{(18)}$$

Substituting from Eqs. 10 and 12*a*, $dx = \frac{(1 - 2 q^2 \sin^2 \phi)(- \delta \sin \phi \, d\phi)}{\sqrt{k^2 (1 - q^2 \sin^2 \phi)(\delta^2 \sin^2 \phi)}}$; or

(selecting $\sqrt{\delta^2} = -\delta$):

$$k\,dx = \frac{(1 - 2\,q^2 \sin^2 \phi)\,d\phi}{\sqrt{1 - q^2 \sin^2 \phi}} \quad \text{(19a)}$$

Eq. 19*a* can be integrated in terms of elliptic integrals, but first some manipulation is necessary. In Eq. 19*a*, the quantity in parentheses can be written: $2\,(1 - q^2 \sin^2 \phi) - 1$, from which:

$$k\,dx = \frac{2\,(1 - q^2 \sin^2 \phi)\,d\phi}{\sqrt{1 - q^2 \sin^2 \phi}} - \frac{d\phi}{\sqrt{1 - q^2 \sin^2 \phi}} \quad \text{(19b)}$$

Integrating Eq. 19*b* from point B to point C, Fig. 1(*a*):

$$\int_0^\lambda k\,dx = k\,\lambda = 2\int_0^{\pi/2} \sqrt{1 - q^2 \sin^2 \phi}\,d\phi - \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - q^2 \sin^2 \phi}} \quad \text{(19c)}$$

In other words,

$$k\,\lambda = 2\,J - K \quad \text{(19d)}$$

in which J and K are elliptic integrals defined by the last two terms of Eq. 19*c*. Substituting $k = \frac{2\,K}{L}$ (see Eq. 13) in Eq. 19*d*, and manipulating:

$$\frac{\lambda}{L} = \frac{J}{K} - \frac{1}{2} \quad \text{(20)}$$

Values of J corresponding to certain values of q are listed in Col. 6, Table 1. The value of J for $\sin^{-1} q = 90°$ can be obtained as follows:

$$J = \int_0^{\pi/2} d\phi\,\sqrt{1 - 1 \sin^2 \phi} = \int_0^{\pi/2} d\phi \cos \phi = \sin \frac{\pi}{2} = 1 \quad \text{(21)}$$

The values of λ/L corresponding to certain values of q, and hence corresponding to certain values of δ/L and $L/\bar{r}$ are shown in Col. 7, Table 1, and are plotted in Fig. 2. Using these values the center and end points of the column can be located on a scale drawing, and rough sketches of the column can thus be drawn for various values of δ/L and $L/\bar{r}$, as shown in Table 2.

The sketches are largely self-explanatory. Case 6 corresponds to a limp string, for which the effective value of $\bar{r}$ is zero.

It should be remembered that the results illustrated by Table 2 do not hold true if the stress exceeds the elastic limit of the material.

Graphical Solution for δ/L and λ/L.—To simplify numerical work, curves will be derived by which the values of δ/L and λ/L can be obtained quickly if the values of E, P/A, and $L/\bar{r}$ are known.

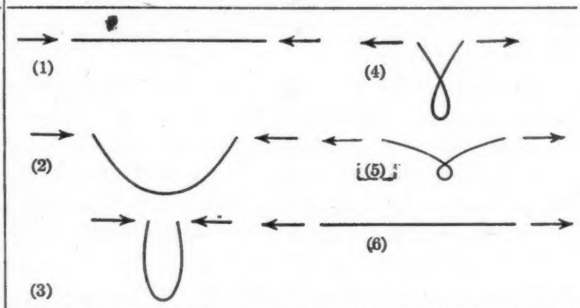
The method used to derive the curve of δ/L versus $L/\bar{r}$ in Fig. 2 can readily be extended to derive the curves of δ/L versus P/A for various values of $L/\bar{r}$. Also, by Fig. 2, the curve of λ/L versus δ/L can be drawn. (The latter curve will presently be shown to be independent of the dimensions of the column.) These curves, shown in Fig. 3, can be used to obtain practically all the information that may be desired about the column. For example, with a given column, if a load P is assumed, δ/L can be determined; hence, the maximum moment,

$P\,\delta$, and the maximum bending stress, $\frac{P\,\delta}{Z}$, can be evaluated (Z is the section modulus). Also, with δ/L determined, the value of λ/L can be selected immediately, and the general shape of the column can be approximated.

TABLE 2.—IDEAL COLUMN SHAPES UNDER VARIOUS CONDITIONS

(E = 30,000,000 LB PER SQ IN. AND P/A = 2 KIPS PER SQ IN.)

Case	$\frac{L}{\bar{r}}$	$\frac{\delta}{L}$	$\frac{\lambda}{L}$	Column shape
1......	384	0	0.50	
2......	415	0.296	0.37	
3......	529	0.401	0.06	
4......	676	0.35	−0.11	
5......	1,330	0.185	−0.316	
6......	∞	0	−0.50	



In selecting a value of λ/L corresponding to a particular value of δ/L, the value of λ/L should lie to the left of the crest of the λ/L-versus-δ/L curve if the value of δ/L lies to the left of the crest of the δ/L-versus-P/A curve, and vice versa.

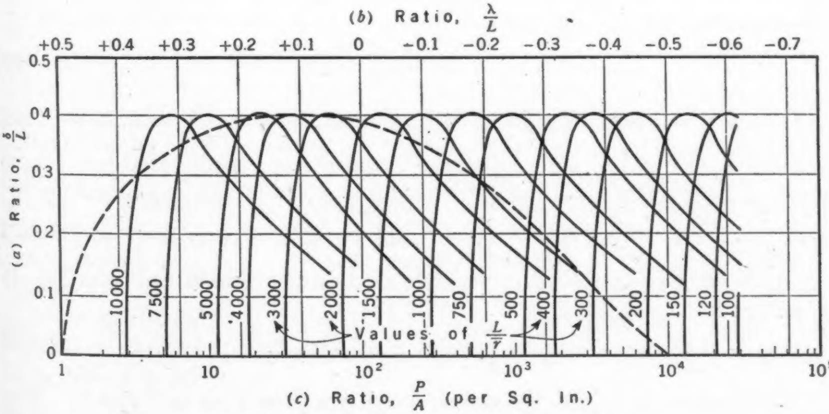


FIG. 3.—GRAPHICAL SOLUTION FOR δ/L AND λ/L

The curves of Fig. 3 are for steel (E = 30,000,000 lb per sq in.). For other values of E, the following relationship should be used:

$$\left(\frac{P}{A}\right)_{\text{indicated}} = \frac{30{,}000{,}000}{E} \times \left(\frac{P}{A}\right)_{\text{actual}} \quad \text{(22)}$$

The use of Fig. 3 can be illustrated by a numerical example. Consider a machine member of spring bronze ($E = 15{,}000{,}000$ lb per sq in.), 24 in. long and 0.1 in. in diameter, D, loaded according to the conditions of Fig. 1(a) with $P = 1.25$ lb. The problem is to find the approximate shape of the elastic curve and the maximum stress. The slenderness ratio is $L/r = \frac{L}{D/4} = \frac{24}{0.1/4} = 960$; and $P/A = \frac{P}{\pi\, D^2/4} = \frac{4 \times 1.25}{\pi\, (0.1)^2} = 160$ lb per sq in. From Fig. 3, $\delta/L = 0.15 \pm$ and $\lambda/L = 0.46$. The shape of the column corresponds roughly to the shape shown as case 2, Table 2. The maximum bending stress is equal to $\frac{P\,\delta}{Z} = \frac{P \times \delta/L \times L}{\pi\, D^3/32} = \frac{32 \times 1.25 \times 0.15 \times 24}{\pi\, (0.1)^3} = 46{,}000 \pm$ lb per sq in.

In this case the direct stress, $P/A = 160$ lb per sq in., is negligible compared to the maximum bending stress.

Exact Shape of the Deflected Column.—To illustrate the completeness of this mathematical solution the exact shape of the elastic curve of a typical column can be derived by finding the values of y/L corresponding to various values of x/L, as x (see Fig. 1(a)) is varied between the values of 0 and λ.

First, integrate Eq. 19b between points B and A, Fig. 1(a), letting $\phi = \beta$ (see Eqs. 10) at point A, as follows:

$$k \int_0^x dx = k\,x = 2 \int_0^\beta d\phi\, \sqrt{1 - q^2 \sin^2 \phi} - \int_0^\beta \frac{d\phi}{\sqrt{1 - q^2 \sin^2 \phi}} \equiv 2\,G - H \quad \text{(23a)}$$

In other words,

$$k\,x = 2\,G - H \quad \text{(23b)}$$

in which G and H are elliptic integrals. From Eq. 13, $k = 2\,K/L$—which value, substituted in Eq. 23b, yields:

$$\frac{x}{L} = \frac{2\,G - H}{2\,K} \quad \text{(24)}$$

Assume that the physical constants are such that $q \equiv \frac{k\,\delta}{2} = \sin 75°$. Then, by Table 1, $K = 2.77$. The constants G and H can be evaluated for different values of β by tables,[5] and the corresponding values of x/L can be determined by Eq. 24. Eq. 16 yields $\frac{\delta}{L} = \frac{q}{K} = \frac{\sin 75°}{2.77} = 0.35$. Therefore, the values of y/L corresponding to the various values of β can be determined from Eqs. 10 by writing:

$$\frac{y}{L} = \frac{\delta}{L} \cos \beta = 0.35 \cos \beta \quad \text{(25)}$$

The computations and the elastic curve are presented in Table 3. Note that the relationship between x/L and y/L depends only on q. This fact leads to an important conclusion, derived as follows:

By referring to Eqs. 13 and 16, it can be seen that q is specified if $\delta/L = q/K$ is specified. Hence, the shape of the column is independent of the column dimensions and depends only on the value of δ/L. (Since there are two load

[5] "A Short Table of Integrals," by B. O. Peirce, Ginn & Co., Boston, Mass., 3d Ed., 1929, pp. 122 and 123.

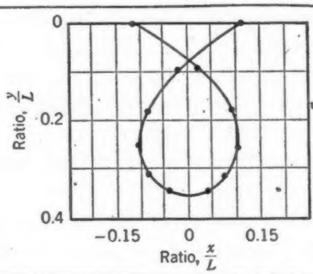
conditions corresponding to each value of δ/L (see Fig. 3), it is necessary to specify whether the column is loaded below or above the crests shown in Fig. 3.)

3. Equilibrium Positions of a Loaded Column

Several difficulties now require attention. First, the fundamental column formula, Eq. 13, is obviously not valid for values of P/A less than the value

TABLE 3.—Exact Shape of the Deflected Column ($q = \sin 75°$)

β	G	H	$\frac{x}{L}$	$\frac{y}{L}$	Elastic curve
0.......	0	0	0	0.35	
15.......	0.259	0.264	0.046	0.338	
30.......	0.502	0.548	0.0822	0.3025	
45.......	0.713	0.873	0.099	0.247	
60.......	0.881	1.28	0.0866	0.1745	
75.......	0.999	1.87	0.0231	0.0904	
90.......	1.08	2.77	−0.11	0	



corresponding to $\delta/L = 0$ in Fig. 3. (The values of P/A and L/r corresponding to $\delta/L = 0$ are termed "critical" values.) For values of P/A below the critical, the solution $y = 0$ is first tested. Substituting the trial solution—

$$y = 0 \text{..........................} \quad (26)$$

—in the fundamental differential Eq. 3, and noting that $\frac{d\theta}{ds}$, the curvature, must equal zero if y is zero everywhere: $P \times 0 \equiv - E\,I \times 0$.

Then, the solution $y = 0$ must be considered a possible solution of Eq. 3 under all admissible conditions. In other words, the column may remain perfectly straight for all values of P/A, whether above or below the critical. Summing up the solutions, Eqs. 13 and 26, the conclusion follows that, with values of P/A less than the critical, the column must remain straight; whereas with values of P/A greater than the critical, the column either can remain straight, or can assume the form dictated by the curves of Fig. 3 and illustrated in Table 2.

It is possible to identify the critical values in this paper with Euler's well-known critical point. By Table 1, the critical value corresponds to $q = 0$, for which value Eq. 13 becomes

$$\frac{k\,L}{2} = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - 0}} = \frac{\phi}{2} \text{..................} \quad (27a)$$

Substituting Eq. 6c in Eq. 27a, $\frac{L}{2}\sqrt{\frac{P}{E\,I}} = \frac{\pi}{2}$; or

$$P = \frac{\pi^2\,E\,I}{L^2} \text{........................} \quad (27b)$$

which is Euler's familiar expression for critical loading of a simple column. The meaning of the word "critical" will be discussed in detail presently.

As demonstrated, when the value of P/A exceeds the critical, there are two widely different solutions for the shape of the column, one corresponding to a perfectly straight column and one to a shape similar to those sketched in Table 2. Do both of these solutions represent a stable condition? To prove mathematically what is already known from experience—that the straight column is unstable, and the bent column stable—proceed as follows:

In any deflected position (see Fig. 1(a)), not necessarily an equilibrium position, the column feels certain internal moments as a result of its curvature. Similarly, the load P applies certain external moments at all points of the deflected column. If, at all points, the internal moment equals the external moment, then the column will be in equilibrium; if the internal moment is greater than the external moment, the curvature will tend to decrease and the column will straighten; and, if the internal moment is less than the external moment, the curvature will tend to increase and the column will assume a sharper bend.

To understand the meaning of the stability of the column, suppose that a loaded column is slightly displaced from its equilibrium position by small transverse forces and that the transverse forces are then removed. If, with the column in the displaced position, the difference between internal and external moments at all points is such as to tend to restore the column to its original position, then the column is stable; otherwise, it is unstable. To simplify the analysis, compare the internal and external moments at the center of the column only. It will then be possible to judge the stability of the column as a whole by inspection.

In evaluating the internal and external moments at the center for different values of δ/L, the internal and external moments at the center are analyzed in units of inch-pounds per unit of cross-sectional area and per unit of length, $\frac{M_i}{A\,L}$ and $\frac{M_e}{A\,L}$, respectively.

First, for a particular value of δ/L, by Fig. 1(a):

$$\frac{M_i}{A\,L} = \frac{P'}{A} \times \frac{\delta}{L} \qquad (28a)$$

in which P'/A is the value of P/A which would hold the column in equilibrium, with δ/L equal to the value chosen. Substituting Eq. 14 in Eq. 13,

$$K = \frac{L\,\sqrt{P'/A}}{2\,r\,\sqrt{E}} \qquad (28b)$$

from which

$$\frac{P'}{A} = \frac{4\,K^2\,E}{(L/r)^2} \qquad (28c)$$

Substituting Eq. 28c in Eq. 28a:

$$\frac{M_i}{A\,L} = \frac{4\,K^2\,E}{(L/r)^2} \times \frac{\delta}{L} \qquad (28d)$$

Also, by Fig. 1(*a*):

$$\frac{M_e}{A\,L} = \frac{P}{A} \times \frac{\delta}{L} \quad \text{(28e)}$$

in which P is the actual load applied to the column.

Consider a typical column subject to various values of the loading, P/A, with E = 30,000,000 lb per sq in.; L/r = 384; and P/A = 1,000, 2,000 (critical), 3,000, and 8,000 lb per sq in. By Eq. 28*d*: $\frac{M_i}{A\,L} = \frac{4\,K^2 \times 30{,}000{,}000}{(384)^2} \times \frac{\delta}{L} = 815\,K^2 \times \frac{\delta}{L}$.

Using methods similar to those for preparing Table 1, the values of $\frac{M_i}{A\,L}$ corresponding to certain values of $\frac{\delta}{L}$ can be tabulated, as shown in Table 4.

TABLE 4.—VALUES OF $\frac{M_i}{A\,L}$

$\sin^{-1} q$ (degrees)	$\delta/L = q/K$	K	$\frac{M_i}{A\,L}$
0	0	1.57	0
30	0.296	1.69	690
60	0.401	2.16	1,520
75	0.35	2.77	2,200
89	0.185	5.43	4,450
90	0	∞	∞

The curve of $\frac{M_e}{A\,L}$ versus $\frac{\delta}{L}$ will be a straight line passing through the origin. Also, if $\frac{\delta}{L}$ = 0.4, then by Eq. 28*e* the values of $\frac{M_e}{A\,L}$ corresponding to the different values of P/A will be as follows:

$\frac{P}{A}$	$\frac{M_e}{A\,L}$
1,000	400
2,000	800
3,000	1,200
8,000	3,200

The values of $\frac{M_i}{A\,L}$ and $\frac{M_e}{A\,L}$ can now be plotted against those of $\frac{\delta}{L}$, as in Fig. 4. The stability of the column, as revealed by these curves, can be studied as follows:

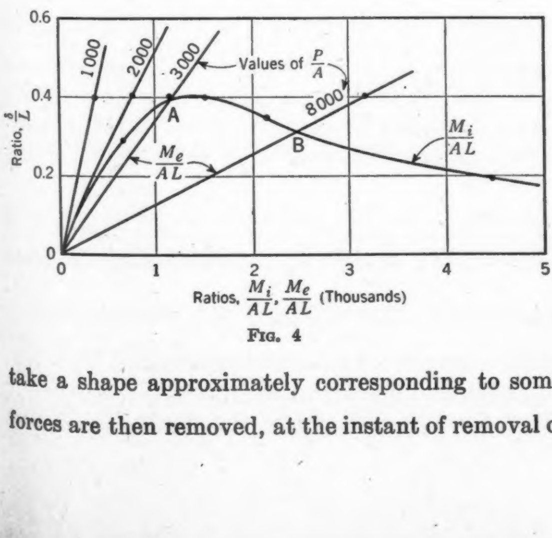


FIG. 4

First, suppose that $\frac{P}{A}$ = 1 kip per sq in., a value that is less than the critical. Assume that the column is in the straight position, $y = 0$. If transverse forces are applied to cause the column to take a shape approximately corresponding to some value of δ/L and if these forces are then removed, at the instant of removal of the forces $\frac{M_i}{A\,L}$ will always

be larger than $\frac{M_e}{A\,L}$, according to Fig. 4, and the column will return to the position $y = 0$. Therefore, with P/A less than the critical value, the column can have only one possible shape—perfectly straight. This result agrees with previous conclusions in this paper.

Next, suppose that $\frac{P}{A} = 3$ kips per sq in., a value that is greater than the critical, and which will cause the column to be in equilibrium at a value of δ/L corresponding to point A, Fig. 4. The shape of the column at this value of δ/L corresponds roughly to case 3, Table 2. If, by transverse forces, the column is slightly displaced in such a direction as to increase δ/L, then $\frac{M_i}{A\,L}$ will be larger than $\frac{M_e}{A\,L}$; and, by comparison with case 3, Table 2, it is seen that the column will return to the equilibrium position as soon as the transverse forces are withdrawn. On the other hand, if the column is displaced in such a direction as to decrease δ/L, then $\frac{M_i}{A\,L}$ will be smaller than $\frac{M_e}{A\,L}$; and, by comparison with case 3, Table 2, it is seen that the column will again return to the equilibrium position when the transverse forces are withdrawn. Point A, therefore, is a point of stable equilibrium.

If $P/A = 8$ kips per sq in., equilibrium point B will lie on the descending section of the $\frac{M_i}{A\,L}$-versus-$\frac{\delta}{L}$ curve of Fig. 4. The shape of the column at point B corresponds roughly to case 4, Table 2. Point B can be proved to be a point of stable equilibrium by the methods just used to investigate point A, but the details of proof will be left to the reader.

Consider that P/A equals either 3 kips per sq in. or 8 kips per sq in., and that the column is in the perfectly straight position. At any value of δ/L less than the equilibrium value at either point A or point B, $\frac{M_e}{A\,L}$ will be larger than $\frac{M_i}{A\,L}$. Therefore, the slightest disturbance of the column will suffice to cause a sudden large deflection corresponding to point A or point B. (The deflection cannot proceed past point A or point B, for reasons discussed in the preceding paragraphs.) This sudden large deflection is called "buckling." It is familiar to engineers, but its mathematical and physical reasons are generally not clearly understood.

Finally, let $P/A = 2$ kips per sq in., the critical value. The curves of $\frac{M_i}{A\,L}$ and $\frac{M_e}{A\,L}$ appear to be tangent at the origin. If this is true (and it is indeed true), then at any point close to the origin $\frac{M_i}{A\,L}$ and $\frac{M_e}{A\,L}$ are nearly equal. Therefore, the deflection of the column must be indeterminate. (This conclusion is proved in an entirely different manner in Section 4.)

The conclusions of this section can be summarized as follows:

(1) At values of P/A below the critical, an ideal column must remain perfectly straight.

(2) At values of P/A above the critical, the column may either remain perfectly straight or may assume a shape dictated by Fig. 3 and illustrated by Table 2. If the column is straight, the slightest displacement will suffice to cause the transverse deflection to change suddenly from zero to the deflection corresponding to the the curves of Fig. 3.

(3) At values of P/A close to the critical, the deflection is indeterminate, and may assume any fairly small value.

It remains to prove the assumption that, at the critical load, the $\frac{M_i}{A\ L}$-versus-$\frac{\delta}{L}$ curve is tangent to the $\frac{M_e}{A\ L}$-versus-$\frac{\delta}{L}$ curve at $\frac{\delta}{L} = 0$. By Eq. 28*e*, the the slope of the $\frac{M_e}{A\ L}$-versus-$\frac{\delta}{L}$ curve at $\frac{\delta}{L} = 0$ equals:

$$\frac{d\left(\frac{M_e}{A\ L}\right)}{d\left(\frac{\delta}{L}\right)} = \frac{d\left(\frac{P_e}{A} \times \frac{\delta}{L}\right)}{d\left(\frac{\delta}{L}\right)} = \frac{P_e}{A} \qquad (29a)$$

in which P_e is the critical load. By Eq. 28*a*, the slope of the $\frac{M_i}{A\ L}$-versus-$\frac{\delta}{L}$ curve equals:

$$\frac{d\left(\frac{P'}{A} \times \frac{\delta}{L}\right)}{d\left(\frac{\delta}{L}\right)} = \frac{P'}{A} + \frac{\delta}{L} \times \frac{d\left(\frac{P'}{A}\right)}{d\left(\frac{\delta}{L}\right)} \qquad (29b)$$

Since at the critical point $\frac{P'}{A} = \frac{P_e}{A}$ and $\frac{\delta}{L} = 0$, then, by comparing Eqs. 29, it may be seen that the curves of $\frac{M_i}{A\ L}$ and $\frac{M_e}{A\ L}$ versus $\frac{\delta}{L}$ are indeed tangent at the origin.

This result leads to a further interesting conclusion in regard to the shape of the curves of Fig. 3 in the region close to the critical point: If $\frac{M_i}{A\ L}$ and $\frac{M_e}{A\ L}$ are equal near the critical point, then δ/L must be indeterminate, as has been stated; if δ/L is indeterminate, it is obvious that $\frac{d\left(\frac{P}{A}\right)}{d\left(\frac{\delta}{L}\right)}$ must be zero at the critical point. This conclusion agrees with the shape of the curves of Fig. 3.

4. Actual Stress and Factor of Safety Below the Critical Point

All structural columns, and most machinery columns, are designed to operate below the critical point. In these columns, the appreciable transverse

deflection that occurs at the critical point constitutes failure. In the case of relatively short columns, the direct stress P/A reaches the yield point and causes failure before the load reaches the critical. The factor of safety, N, must therefore be based either on P_c, or on P_y (the load that produces a direct stress equal to σ_y, the yield-point stress)—whichever governs.

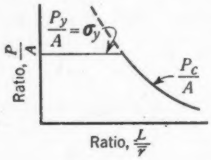


Fig. 5

In the case of steel columns, with E and σ_y taken at 30,000,000 lb per sq in. and 30,000 lb per sq in., respectively, the values of P_y/A and P_c/A may be plotted against the values of $L/\bar{r}$ (see Fig. 5). The ratio P_y/A = a constant = σ_y, whereas P_c/A may be derived from Euler's formula by writing $P_c = \frac{\pi^2 E I}{L^2} = \frac{\pi^2 E A \bar{r}^2}{L^2}$; or

$$\frac{P_c}{A} = \frac{\pi^2 \times 30{,}000{,}000}{\left(\frac{L}{\bar{r}}\right)^2} = \frac{296{,}000{,}000}{\left(\frac{L}{\bar{r}}\right)^2} \qquad (30)$$

The dotted section of the curve in Fig. 5 would be reached only after failure had already occurred, caused by a direct stress greater than σ_y. Neglecting this section, one may proceed to redraw the remainder of the curve for different factors of safety, simply dividing the values of P_y/A and P_c/A by the assumed values of N. In Fig. 6, if σ_y is not equal to 30,000 lb per sq in., or, if E is some value other than 30,000,000 lb per sq in., $N_y \propto \sigma_y$ and $N_c \propto E$.

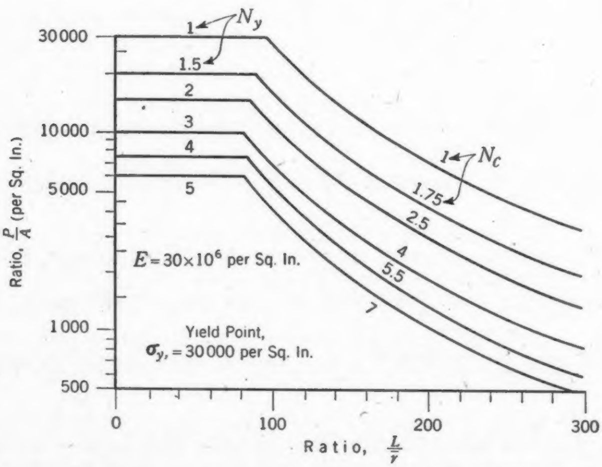


Fig. 6.—Factors of Safety of Simple Ideal Columns (If $\sigma_y \neq 30{,}000$ Lb per Sq In. or $E \neq 30{,}000{,}000$ Lb per Sq In., Then $N_y \propto \sigma_y$ and $N_c \propto E$)

It seems fair to assign larger values of N to the part of each curve based on P_c than to the part based on P_y, since failure at the critical point is much more violent and complete than is failure caused by the direct stress reaching the value σ_y. If the quantity $N - 1$ is considered as a "margin of safety," it seems

desirable that:

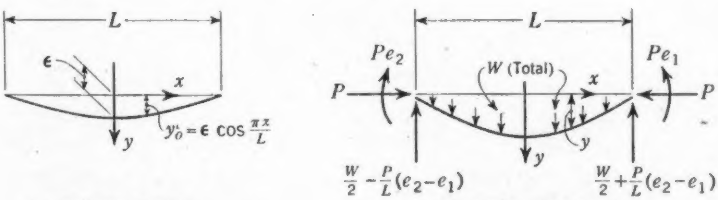
$$N_c - 1 = 1.5\,(N_y - 1)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(31)$$

in which the subscripts c and y denote whether the factor of safety is based on P_c or P_y. Accordingly, the values of N that are indicated for the straight and curved parts of each of the curves of Fig. 6 were chosen to agree with Eq. 31.

Fig. 6 enables the designer to determine the factor of safety of any simple, ideal column loaded below the critical point. According to the analysis in Section 3, the stress in such a column is nothing more or less than the direct stress. The actual column, however, differs from the ideal in certain important respects:

End Restraint.—The actual column is usually subjected to a certain amount of end restraint. This condition tends to strengthen the column, and may be taken advantage of in design when an appreciable degree of end restraint may be expected. In practice, however, it is often difficult to evaluate the amount of end restraint. It is shown in standard texts that the effect of perfect end restraint is to reduce the effective $L/\bar{r}$-value by half.

External Moments and Irregularities of Shape.—External moments in addition to $P\,y$ (see Fig. 1(*a*)) may be applied to the column, because of eccentricity of the load P or because of forces applied anywhere on the column, or because of the weight of the column itself if it is not disposed vertically. The moment exerted at any point of the column by these external moments will be called $M(x)$, the notation signifying that M is a function of x. (For reasons of simplicity, external moments other than $P\,y$, which are functions of y, will not be considered.) Also, the column may have some initial irregularity of shape. Let y_o equal the initial deflection at any point.



(*a*) INITIAL SHAPE (*b*) LOADED CONDITIONS

FIG. 7

An example of a column such as the one under discussion is shown in Fig. 7, in which W is the total load uniformly distributed.

Only columns whose transverse deflection is small compared to their length are considered in this section. With such columns, the following simplifications are possible:

(*a*) The curvature may be expressed by $\frac{d^2y}{dx^2}$ (see Section 1, Eq. 4).

(*b*) The projected length of the column on the x-axis equals the developed length.

Eq. 4 can be extended to mean that M, the moment at any point, is equal to $-E\,I$ times the difference between the final and initial curvatures of the

column. Then, since the initial curvature of the column is equal to $\frac{d^2y_o}{dx^2}$, Eq. 4 can be rewritten:

$$M = P\,y + M(x) = -E\,I\left(\frac{d^2y}{dx^2} - \frac{d^2y_o}{dx^2}\right) \qquad (32)$$

Letting $k^2 \equiv \frac{P}{E\,I}$, as before, Eq. 32 can be written:

$$\frac{d^2y}{dx^2} + k^2\,y = \frac{d^2y_o}{dx^2} - \frac{M(x)}{E\,I} \qquad (33)$$

Eq. 33 is a linear differential equation of the second order with constant coefficients, and is treated in every standard work on differential equations. For the column shown in Fig. 7, the solution of Eq. 33 works out to be:

$$y = \left(\frac{W}{P\,L\,k^2} + \frac{e_2 + e_1}{2}\right)\frac{\cos k\,x}{\cos\frac{k\,L}{2}} - \frac{e_2 - e_1}{2}$$

$$\times \frac{\sin k\,x}{\sin\frac{k\,L}{2}} + \frac{\epsilon\,\pi^2}{\pi^2 - k^2\,L^2}\cos\frac{\pi\,x}{L} + \frac{W\,x^2}{2\,L\,P}$$

$$+ \frac{e_2 - e_1}{L}\,x - \frac{W}{P}\left(\frac{L}{8} + \frac{1}{L\,k^2}\right) - \frac{e_2 + e_1}{2} \qquad (34)$$

Knowing y, the moment M can be determined by Eq. 32. Hence, the maximum bending stress at any section, $\sigma_b = \frac{M\,c}{I}$ (in which c is the distance from the neutral axis to the extreme fibers), can be calculated. The total stress is:

$$\sigma_t = \sigma_b + \frac{P}{A} \qquad (35a)$$

and its maximum value can be determined by setting $\frac{d\sigma_t}{dx} = 0$, according to the routine methods of the differential calculus. The factor of safety may be defined as the ratio between the external loading producing the yield-point stress σ_y and the actual external loading, and may be determined by trial.

If $e_2 = e_1 = e$, and $\epsilon = W = 0$, then, by Eqs. 34 and 32, it can readily be shown that the maximum bending stress equals:

$$\sigma_b = E\,k^2\,e\,c\sec\frac{k\,L}{2} \qquad (35b)$$

Eq. 35*b* is the well-known secant formula.

Again, if $e_2 = e_1 = W = 0$, then, by Eqs. 34 and 32, the maximum bending stress equals:

$$\sigma_b = \frac{E\,\epsilon\,c\,\pi^2\,k^2}{\pi^2 - k^2\,L^2} \qquad (35c)$$

Eq. 35*c* should prove useful in many practical cases. Most structural members are not fabricated perfectly straight, but have a camber. The actual

shape of the column is usually roughly similar to a cosine curve, and the value of ϵ can be determined easily by stretching a string between the ends of the member. By substituting the highest probable value of ϵ in Eq. 35*c*, the highest probable bending stress caused by initial camber may be determined.

Eq. 35*c*, the camber formula, should be carefully distinguished from Eq. 35*b*, the secant formula, which gives the bending stress due to eccentricity of the load P.

It is possible to derive Euler's formula for critical loading from Eq. 34 because, if $e_2 = e_1 = W = \epsilon = 0$, then:

$$y = \frac{\epsilon\,\pi^2}{\pi^2 - k^2\,L^2} \quad \text{(36)}$$

If ϵ is reduced to zero and $k^2 = \frac{\pi^2}{L^2}$, then $y = \frac{0}{0}$, which is indeterminate, and $P = \frac{\pi^2\,E\,I}{L^2}$, which constitutes Euler's condition.

In the case just mentioned, if $\epsilon = 0$ and $k^2 \neq \frac{\pi^2}{L^2}$, then Eq. 34 reduces to $y = 0$. Therefore, the simplified differential Eq. 4, from which Eq. 34 was derived, fails to give a satisfactory solution in the case of an ideal column. For that reason it was necessary to use the more exact, if more difficult, Eq. 3 in developing the theory of the ideal column.

The Step-by-Step Method.—The formal solution of Eq. 33, tedious under the best of conditions, becomes prohibitively complicated in certain cases, especially when I is variable or $M(x)$ is discontinuous. In such cases it is usually more practical to use some form of numerical integration. The step-by-step method of numerical integration, a mathematical weapon of long standing, has certain distinct advantages in connection with the solution of column problems:

(*a*) It is not a trial-and-error method, but leads directly to the solution.

(*b*) It is universally applicable to all columns whose relative deflection is small, enabling the designer to solve widely different types of column problems with only minor changes of technique.

(*c*) It may be used to obtain a rough solution fairly quickly or to obtain an accurate solution at the expense of more numerical work, as desired.

In the step-by-step method the column is divided into a number of intervals, not necessarily equal. All variable quantities are considered to be constant during each interval. The change, Δ, in any quantity in each interval is obtained numerically, and the value of any quantity at the end of any interval is obtained by adding Δ to the value of the quantity at the beginning of the interval.

Applying this process of reasoning to Eq. 32 and letting the order of differentiation be denoted by primes $\left(\text{for example, } y'' = \frac{d^2y}{dx^2}\right)$:

$$y'' = y''_o - \frac{M}{E\,I} \quad \text{(37)}$$

Integrate Eq. 37 from x = A to an indefinite point x = B, considering that the values of y and of its derivatives are known at point A. Then:

$$y'_B - y'_A = (y'_o)_B - (y'_o)_A - \int_A^B \frac{M\,dx}{E\,I} \qquad \text{(38a)}$$

Now, if the interval is small, $\frac{M}{E\,I}$ can be considered to be practically constant between points A and B. Then:

$$\Delta y' = \Delta y'_o - \frac{i\,M}{E\,I} \qquad \text{(38b)}$$

in which $i = x_B - x_A$.

Interpreted physically, Eq. 38*b* means that the change of slope of the elastic curve between point A and point B equals the change of slope of the unloaded column, plus the change of slope caused by moment—the latter term being equal to $-\frac{i\,M}{E\,I}$.

Next, integrate Eq. 38*a* over the interval, once more letting $(y'_o)_B = y'_o$ and $y'_B = y'$, remembering that $(y'_o)_A$ and y'_A are constant:

$$y_B - y_A - i\,y'_A = (y_o)_B - (y_o)_A - i\,(y'_o)_A - \int_A^B \int_A^B \frac{M\,dx\,dx}{E\,I} \qquad \text{(39)}$$

If $x_B = x$ and $\frac{M}{E\,I}$ is a constant during the interval, the last term of Eq. 39 can be expanded as follows:

$$\int_A^B \int_A^B \frac{M\,dx\,dx}{E\,I} = \frac{M}{E\,I}\int_A^B (x - x_A)\,dx = \frac{M}{E\,I}\left(\frac{x^2}{2} - x_A\,x\right)_A^B$$

$$= \frac{M}{E\,I}\left(\frac{x^2_B}{2} - \frac{x^2_A}{2} - x_A\,x_B + x^2_A\right) = \frac{M}{2\,E\,I}(x_B - x_A)^2 = \frac{i^2\,M}{2\,E\,I} \qquad \text{(40)}$$

Thus, Eq. 39 becomes:

$$\Delta y = \Delta y_o - i\,(y'_o)_A + i\,y'_A - \frac{i^2\,M}{2\,E\,I} \qquad \text{(41)}$$

Then, in physical terms, the change of deflection over an interval equals the change of deflection of the unloaded column, plus the change of deflection produced by the slope $y'_A - (y'_o)_A$ acting over the length i, plus the change of deflection caused by moment—the last term being equal to $-\frac{i^2\,M}{2\,E\,I}$.

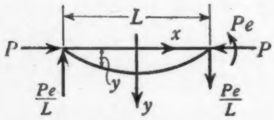


FIG. 8

Eqs. 38*b* and 41 are the keys to the method. The actual technique of procedure will be illustrated by a numerical example. The column shown in Fig. 8 is a steel rod 1 in. in diameter. The unloaded deflection may be expressed by

$$y_o = \epsilon \cos \frac{\pi\,x}{L} \qquad \text{(42)}$$

Numerical values are as follows: $E = 30{,}000{,}000$ lb per sq in.; $I = 0.0491$ in.^{4}; $A = 0.785$ in.^{2}; $L = 30$ in.; $\epsilon = 0.1$ in.; $P = 2{,}700$ lb; and $e = 1$ in.

The column will be divided into 5-in. intervals, starting from the left end. Considering Eq. 38*b*, it is clear that y'_o and $\dfrac{i\,M}{E\,I}$ must be evaluated. The first of these terms can be calculated from the known initial deflection of the column, as follows: $y_o = \epsilon \cos \dfrac{\pi\,x}{L}$, $y'_o = -\dfrac{\epsilon\,\pi}{L} \sin \dfrac{\pi\,x}{L} = -0.01048 \sin (6\,x)^\circ$.

If $(y'_o)_n$ represents the value of y'_o at the beginning of the *n*th interval, then:

$$\Delta y'_o = (y'_o)_{n+1} - (y'_o)_n \qquad (43)$$

The calculations leading to the values of $\Delta y'_o$ are tabulated in Cols. 1, 2, 3, and 4 of auxiliary Table 5(*a*), and the values of $\Delta y'_o$ are listed in Col. 15 of main Table 5(*b*).

In cases where $\Delta y'_o$ cannot be derived analytically, it can always be obtained graphically, from the known initial deflection of the column.

Next consider the quantity $\dfrac{i\,M}{E\,I}$:

$$\frac{i\,M}{E\,I} = \frac{i\,M(x)}{E\,I} + \frac{i\,M(y)}{E\,I} \qquad (44)$$

in which $M(x)$ is a function of x and constants, and $M(y)$ is a function of y alone. Moment $M(x)$ can be calculated immediately, by the same methods that are used to calculate the moment at any point of an ordinary beam, whereas the calculation of $M(y)$ at any point must be deferred until the value of y at that point is known.

A word in regard to sign conventions will be in order at this point. If x is positive to the right, and y is positive downward, as in Fig. 8, then moments acting to the left of any section are positive if they are clockwise, and moments acting to the right of the section are positive if they are counterclockwise. This condition will be found to be in accordance with previous work in this paper. By Fig. 8:

$$M(x) = \frac{P\,e}{L}\left(\frac{L}{2} + x\right) = \frac{P\,e}{2} + \frac{P\,e\,x}{L} \qquad (45a)$$

Then:

$$\frac{i\,M(x)}{E\,I} = \frac{5}{30 \times 10^6 \times 0.0491}\left(\frac{2{,}700}{2} + \frac{2{,}700\,x}{30}\right) = 0.00459 + 0.000306\,x \qquad (45b)$$

In calculating the value of $\dfrac{i\,M(x)}{E\,I}$, the value of x at the middle of the interval will be used. The value of $\dfrac{i\,M(x)}{E\,I}$ so obtained will be a good approximation of the average value of $\dfrac{i\,M(x)}{E\,I}$ over the entire interval. (The subscript i will be used where necessary to denote quantities which are taken at the middle

of an interval.) The calculations for determining $\frac{i\,M(x)}{E\,I}$ are tabulated in Cols. 8, 9, and 10, Table 5.

The next step is to compute the value of $\frac{i\,M(y)}{E\,I}$. By Fig. 8, $M(y) = P\,y$, or

$$\frac{i\,M(y)}{E\,I} = \frac{5}{30 \times 10^6 \times 0.0491} \times 2{,}700\,y = 0.00918\,y \qquad (46a)$$

An approximate value y_i (the value of y at the middle of any interval) can be determined for use in Eq. 46*a*; thus:

$$y_i = y + \frac{i}{2}\,y' \qquad (46b)$$

in which the quantities without subscripts are taken at the beginning of the interval. From Fig. 8, at the beginning of the first interval, $y = 0$ (Col. 23)

TABLE 5.—COMPUTATIONS

x (1)	$6\,x$ (2)	Sin $(6\,x)^0$ (3)	y'_o (4)	y_o (5)	Δy_o (6)	$-i\,y'_o$ (7)	x_i (8)	$0.000306\,x_i$ (9)	$\frac{i\,M(x)}{E\,I}$ (10)	$\frac{i\,M(y)}{E\,I}$ (11)	y_i (12)	y (13)
(*a*) AUXILIARY TABLE												
−15	−90	−1	0.01048	0	0.05	−0.0524	−12.5	−0.003824	0.000766	$0.02295\,\theta$	$2.5\,\theta$	0
−10	−60	−0.866	0.00908	0.05	0.0366	−0.0454	− 7.5	−0.002295	0.002295	-0.000089 $+0.0677\,\theta$	$-\ 0.009726$ $+\ 7.386\,\theta$	0.115
− 5	−30	−0.5	0.00524	0.0866	0.0134	−0.0262	− 2.5	−0.000765	0.003825	-0.000459 $+0.1096\,\theta$	$-\ 0.049976$ $+11.929\,\theta$	0.205
0	0	0	0	0.1	−0.0134	0	2.5	0.000765	0.005355	-0.001228 $+0.146\,\theta$	$-\ 0.133756$ $+15.930\,\theta$	0.245
5	30	0.5	−0.00524	0.0866	−0.0366	0.0262	7.5	0.002295	0.006885	-0.002433 $+0.176\,\theta$	$-\ 0.265076$ $+19.200\,\theta$	0.225
10	60	0.866	−0.00908	0.0500	−0.05	0.0454	12.5	0.003824	0.008414	-0.004025 $+0.198\,\theta$	$-\ 0.438246$ $+21.59\,\theta$	0.142
15	90	1	−0.01048	0	..	..	..	..	..	..	..	..

and $y' = \theta$ (Col. 18), in which θ is defined as the angle of slope at the left end of the column. Then $\frac{i\,y'}{2} = 2.5\,\theta$ (Col. 24), and $y_i = 0 + 2.5\,\theta$ (Col. 12). By Eq. 46*a*, $\frac{i\,M(y)}{E\,I} = 0.00918 \times 2.5\,\theta = 0.02295\,\theta$ (Col. 11). Adding Cols. 10 and 11: $\frac{i\,M}{E\,I} = 0.000766 + 0.02295\,\theta$ (Col. 16). By Eq. 38*b*, $\Delta y' = -\ 0.002166$

$-0.02295\,\theta$ (Col. 17). Adding Cols. 17 and 18, the value of y' at the beginning of the second interval is obtained; thus: $y' = -0.002166 + 0.97705\,\theta$.

It will be necessary to carry along the unknown, θ, in the computations until some known condition of the column makes a solution possible. For the present, θ will be treated exactly as any other of the values.

The next step is the utilization of Eq. 41. The value of Δy_o is equal to:

$$(y_o)_{n+1} - (y_o)_n \qquad (47a)$$

and may be obtained analytically or graphically. In the present case,

$$y_o = 0.1 \cos (6\,x)^\circ \qquad (47b)$$

The values of y_o and Δy_o are listed in Cols. 5 and 6, Table 5. The values of y'_o have already been derived (Col. 4). Since $i = 5$, the values of $-i\,y'_o$ (Col. 7) can be obtained. Then the values of $-\dfrac{\Delta y_o}{i\,y'_o}$ are listed in Col. 20.

FOR THE COLUMN IN FIG. 8

y_o (14)	$\Delta y'_o$ (15)	$\frac{i\,M}{E\,I}$ (16)	$\Delta y'$ (17)	y' (18)	$i\,y'$ (19)	$\frac{\Delta y_o}{-i\,y'_o}$ (20)	$-\frac{i^2 M}{2\,E\,I}$ (21)	Δy (22)	y (23)	$\frac{i\,y'}{2}$ (24)	x (1)
					(*b*) MAIN TABLE						
0	−0.00140	0.000766 +0.02295 θ	−0.002166 −0.02295 θ	θ	5 θ	−0.0024	−0.001916 −0.0574 θ	−0.004316 +4.9426 θ	0	2.5 θ	−15
0.114	−0.00384	0.002206 +0.0677 θ	−0.006046 −0.0677 θ	−0.002166 +0.97705 θ	−0.01082 +4.885 θ	−0.0088	−0.00551 −0.169 θ	−0.025130 +4.716 θ	− 0.004316 + 4.9426 θ	−0.005410 +2.443 θ	−10
0.203	−0.00524	0.003366 +0.1096 θ	−0.008606 −0.1096 θ	−0.008212 +0.9094 θ	−0.04105 +4.545 θ	−0.0128	−0.00841 −0.274 θ	−0.06226 +4.271 θ	− 0.029446 + 9.659 θ	−0.02053 +2.270 θ	− 5
0.244	−0.00524	0.004127 +0.146 θ	−0.009367 −0.146 θ	−0.016818 +0.7998 θ	−0.0842 +4.00 θ	−0.0134	−0.01032 −0.365 θ	−0.10792 +3.635 θ	− 0.091706 +13.930 θ	−0.04205 +2.00 θ	0
0.224	−0.00384	0.004452 +0.176 θ	−0.008292 −0.176 θ	−0.026185 +0.654 θ	−0.1310 +3.270 θ	−0.0104	−0.01112 −0.440 θ	−0.15252 +2.830 θ	− 0.199626 +17.565 θ	−0.06545 +1.635 θ	5
0.141	−0.00140	0.004389 +0.198 θ	..	−0.034477 +0.478 θ	−0.1723 +2.390 θ	−0.0046	−0.01098 −0.495 θ	−0.18788 +1.895 θ	− 0.352146 +20.395 θ	−0.0861 +1.195 θ	10
..	..	..	..	..	..	..	..	..	− 0.540026 +22.290 θ	..	15

The first value of $i\,y' = 5\,y'$ can be obtained from Col. 18, and is shown in Col. 19. Also, the value of $-\dfrac{i^2 M}{2\,E\,I} = -2.5 \times \dfrac{i\,M}{E\,I}$ can be obtained from Col. 16, and is shown in Col. 21. Referring to Eq. 41, Δy is equal to $-0.004316 + 4.9426\,\theta$ (Col. 22). Adding this value to the first value of y (Col. 23), the value of y at the beginning of the second interval is finally obtained; thus: $y = -0.004316 + 4.9426\,\theta$.

The reader will have noted that care was exercised in grouping and arranging the columns properly for easy numerical manipulation. The arrangement actually chosen was not based on any general rules, but was built up after a few trial calculations to suit the needs of this particular problem.

The calculations will now be traced through the second interval. Knowing y (Col. 23) and y' (Col. 18) from previous calculations, the values of y_i (Eq. 46*b* and Col. 12) and of $\frac{i\,M(y)}{E\,I}$ (Eq. 46*a* and Col. 11) can be found. Cols. 10 and 11 are added to find $\frac{i\,M}{E\,I}$ (Col. 16). By Eq. 38b, $\Delta y'$ (Col. 17) is found. From previous computation, $y' = -\,0.002166 + 0.97705\,\theta$ (Col. 18). Adding this value to the second value of $\Delta y'$, $y' = 0.008212 + 0.9094\,\theta$ at the beginning of the third interval.

The second values of $i\,y'$ (Col. 19) and $-\frac{i^2\,M}{2\,E\,I}$ (Col. 21) are easily obtained from Cols. 18 and 16, respectively. By Eq. 41 $\Delta y = -\,0.02513 + 4.716\,\theta$ (Col. 22). Adding this value to the second value of y results in: $y = -\,0.029446 + 9.659\,\theta$ at the beginning of the third interval (Col. 23).

Thus, the calculations are carried along until, finally, $y = -\,0.540026 + 22.29\,\theta$ at $x = 15$; but, according to Fig. 8, y at this point equals zero. Therefore, $0 = -\,0.540026 + 22.29\,\theta$, or $\theta = 0.0242$.

With this value of θ substituted in the computations, any numerical values desired can be determined. For example, the numerical values of y can be computed, and are shown in Col. 13. For the sake of comparison, the values of y_e in Col. 14 are the values of y obtained by the "exact" equation (Eq. 34). The correspondence is remarkably close, in spite of the fact that the calculations were performed on an ordinary 10-in. slide rule. The discrepancy is less than 1%, an insignificant amount in engineering calculations.

With the value of θ known, the values of $\frac{i\,M}{E\,I}$ (Col. 16) can be determined. The maximum moment is found to occur at the right end of the column, and is found equal simply to $P\,e = 2{,}700$ in.-lb. The maximum bending stress is $\sigma_b = \frac{M\,c}{I} = \frac{2{,}700 \times 0.5}{0.0491} = 27{,}500$ lb per sq in. The direct stress, P/A, equals 2,700/0.785 = 3,500 lb per sq. in.; and the total stress equals $\sigma_t = \sigma_b + \frac{P}{A} = 31{,}000$—say, 30,000 lb per sq in.

Using a factor of safety of $N = 4$ based on a yield point of 30,000 lb per sq in., the allowable load is found equal to $P_a = \frac{P}{4} = \frac{2{,}700}{4} = 675 \pm$ lb in which P_a is the allowable load.

This result can be compared with the result yielded by one of the rough formulas commonly used by engineers in designing columns subject to both axial load and moment. Such a formula is:

$$\left(\frac{P}{A}\right)_a = \frac{P}{A} + \frac{M\,c}{I} \qquad (48)$$

in which P/A is the direct stress, M is the maximum moment with the column at its initial deflection, and $\left(\frac{P}{A}\right)_a$ is the direct stress that would be allowed for the column if M were zero. Noting that $\frac{L}{r} \doteq 120$, the proper value of $\left(\frac{P}{A}\right)_a$ to yield a factor of safety of $N = 4$ is 5,150 lb per sq in., by Fig. 6. The value of M can be shown to be a maximum at the right end of the column, where it is equal to $P\,e = P$. Then, by Eq. 48: $5{,}150 = \frac{P}{0.785} + \frac{P \times 0.5}{0.0491}$, or $P = 450$ lb.

Then, for a factor of safety of $N = 4$, the allowable value $P = 450$ lb, as obtained by the approximate formula (Eq. 48), is 33% smaller than the allowable value $P = 675$ lb, as obtained by the relatively accurate step-by-step method. Clearly, formulas which are of the same order of approximation as Eq. 48 are unsatisfactory in close design. They are very useful, however, as first approximations.

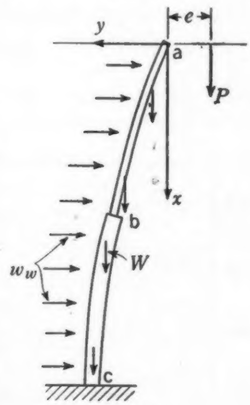


FIG. 9

The versatility of the step-by-step method can be demonstrated by applying it to a very different type of problem. The column shown in Fig. 9 is fixed at its lower end and free at its upper end. The lower part, 200 in. long, is made of steel pipe (outside diameter, 4 in. and inside diameter, 3 in.). The upper part, 250 in. long, is made of steel pipe (outside diameter, 3 in. and inside diameter, 2 in.). A wind force, of 30 lb per square foot of projected area, acts transversely. A load P acts at the top with an eccentricity of e in. The distributed weight of the column acts vertically.

The initial deflection of the column is assumed to be practically zero, so that Eqs. 38*b* and 41, respectively, reduce to:

$$\Delta y' = -\frac{i\,M}{E\,I} \quad \text{(49a)}$$

and

$$\Delta y = i\,y'_A - \frac{i^2\,M}{2\,E\,I} \quad \text{(49b)}$$

For illustrative purposes the column is divided into only four intervals. Let $i = 125$ in. between points a and b, and 100 in. between points b and c. Also, let the wind force be represented by w_w pounds per linear inch, and assume that W, the weight of the column in each interval, is concentrated at the middle of the interval. (This last simplification is convenient rather than necessary; a closer approximation to the true condition could readily be obtained, if desired.)

The axes x and y have been chosen to agree with the adopted sign conventions. By rotating Fig. 9 through 90° counterclockwise, it can be seen that this column is really one half of a symmetrical column of the conventional type, with the left end at point a and the center at point c. Begin the computation

at point a, where $y = 0$ and $y' = \theta$; then progress along the column as far as point c. At this point the knowledge that $y' = 0$ will permit a solution for θ. Knowing the value of θ, it is then possible to evaluate y, y', M, etc., at the various points of the column.

The numerical values are as follows: $P = 1{,}000$ lb; $e = 10$ in.; and $E = 30{,}000{,}000$ lb per sq in.

From point a to point b: $A = 3.93$ in.^{2}; $I = 3.19$ in.^{4}; $w_w = \frac{30 \times 3}{144} = 0.625$ lb per in.; $i = 125$ in.; $W = i\,A \times$ weight per cubic inch $= 125 \times 3.93 \times 0.2834 = 140$ lb; and $\frac{i}{E\,I} = 1.305 \times 10^{-6}$.

TABLE 6.—COMPUTATION

x (1)	x_i (2)	$0.3125\, x_i^2$ (a) (3)	$-52\, x_i$ (4)	$W\, y_i$ (5)	ΣW (6)	$P\, y_i$ (7)	$y_i\, \Sigma W$ (8)	$\Sigma(W\, y_i)$ (9)	y_i (10)	$M(x)$ (11)
(a) AUXILIARY TABLE										
0	62.5	1,222	..	8,750 θ	140	62,500 θ	8,750 θ	8,750 θ	62.5 θ	11,220
125	187.5	10,980	..	−256.5 +24,800 θ	280	−1,832 +177,300 θ	−513 +49,600 θ	−256.5 +33,550 θ	−1.832 +177.3 θ	20,980
250	300	37,500[a]	−15,600	−976 +39,400 θ	436	−6,253 +252,300 θ	−2,730 +110,100 θ	−1,232 +72,950 θ	−6.253 +252.3 θ	38,400
350	400	66,600[a]	−20,800	−1,772 +48,000 θ	592	−11,370 +307,600 θ	−6,730 +182,000 θ	−3,004 +121,000 θ	−11.37 +307.6 θ	62,300
450	..	..	..	.. ..	..	.. ..	.. ..	.. ..		..

[a] For x = 250 and 350,

From point b to point c: $A = 5.5$ in.^{2}; $I = 8.59$ in.^{4}; $w_w = \frac{30 \times 4}{144} = 0.833$ lb per in.; $i = 100$ in.; $W = 100 \times 5.5 \times 0.2834 = 156$ lb; and $\frac{i}{E\,I} = 0.388 \times 10^{-6}$.

Also:

$$M(x)\Big]_a^b = P\,e + \frac{w_w\,x^2}{2} = 10{,}000 + 0.3125\,x^2 \qquad (50a)$$

and

$$M(x)\Big]_b^c = 10{,}000 + 250 \times 0.625\,(x - 125) + 0.833\,\frac{(x - 250)^2}{2}$$

$$= 0.4165\,x^2 - 52\,x + 16{,}500 \qquad (50b)$$

The expression for the value of $M(y)$ at the middle of the nth interval is derived as follows, remembering that the subscript i refers to values taken at the middle of any interval: $[M(y)]_n = P(y_i)_n + W_{n-1}[(y_i)_n - (y_i)_{n-1}] + W_{n-2}[(y_i)_n - (y_i)_{n-2}] + \cdots$; or

$$[M(y)]_n = P(y_i)_n + (y_i)_n \sum_1^{n-1} W - \sum_1^{n-1} W\, y_i \ldots\ldots\ldots\ldots (51)$$

Adding the quantity $0 = W_n(y_i)_n - W_n(y_i)_n$ to the right-hand side of Eq. 51:

$$M(y) = P\, y_i + y_i \sum W - \sum W\, y_i \ldots\ldots\ldots\ldots\ldots (52)$$

in which the quantities under the summation signs are to be taken from 1 to n.

...OR THE COLUMN IN FIG. 9

$M(y)$ (12)	M (13)	$\Delta y' = \frac{-i\,M}{E\,I}$ (14)	y' (15)	$i\,y'$ (16)	$\frac{-0.5\,i^2\,M}{E\,I}$ (17)	Δy (18)	y (19)	$0.5\,i\,y'$ (20)	x (1)
(b) MAIN TABLE									
62,500 θ	11,220	−0.01465	θ	125 θ	−0.916	−0.916	0	62.5 θ	0
	+62,500 θ	−0.0816 θ			−5.10 θ	+119.9 θ			
-2,088	18,890	−0.02465	−0.01465	−1.83	−1.542	--3.372	−0.916	−0.916	125
+193,350 θ	+193,350 θ	−0.2525 θ	+0.9184 θ	+114.8 θ	−15.79 θ	+99.0 θ	+119.9 θ	+57.4 θ	
-7,751	30,650	−0.0119	−0.03930	−3.93	−0.595	−4.525	−4.288	−1.965	250
+289,450 θ	+289,450 θ	−0.1122 θ	+0.6659 θ	+66.59 θ	−5.61 θ	+60.98 θ	+218.9 θ	+33.35 θ	
-15,100	47,200	−0.01832	−0.0512	−5.12	−0.916	−6.036	−8.813	−2.56	350
+368,600 θ	+368,600 θ	−0.1431 θ	+0.5537 θ	+55.37 θ	−7.160 θ	+48.21 θ	+279.9 θ	+27.65 θ	
..	..	..	−0.0695	..	..	..	−14.849	..	450
..	..	..	+0.4106 θ	..	..	..	+328.1 θ	..	

...these values are 0.4165 x^2_i.

The calculations are shown in auxiliary Table 6(*a*) and in main Table 6(*b*). Cols. 1 through 4, Table 6(*a*), lead to the values of $M(x)$ listed in Col. 11. To evaluate $M(y)$, it is first necessary to evaluate y_i. Knowing that $y = 0$ (Col. 19) and $y' = \theta$ (Col. 15) at $x = 0$, it is possible to obtain the first value of $y_i = y + \frac{i}{2} y'$ (Cols. 20 and 10). Next, since the different values of W are known, it is easy to tabulate the values of $\sum W$ (Col. 6). Then, the first values of $y_i\,(\sum W)$ (Col. 8), $W\,y_i$ (Col. 5), and $\sum W\,y_i$ (Col. 9) can be listed. Also, the first value of $P\,y_i$ can be listed (Col. 7). Then, by Eq. 52, the first value of $M(y)$ can be calculated (Col. 12). By adding this value to $M(x)$ (Col. 11), the first values of M (Col. 13) and of $\Delta y' = -\frac{i\,M}{E\,I}$ (Eq. 49*a* and Col. 14) can be listed. The latter value is added to the first value of y' (Col. 15) to determine the second value of y'.

Next, the value of Δy is calculated by Cols. 16 and 17 (see Eq. 49*b*), and is listed in Col. 18. This value is added to the first value of y (Col. 19) to obtain the second value of y. The second value of $y_i = y + \frac{i}{2} y'$ can be computed using the second values of y and y'. The computations then proceed as before.

Finally, at $x = 450$, y' is found to equal $-0.0695 + 0.4106\,\theta$. Since this quantity is known to be equal to zero, θ can be evaluated, and is found to equal 0.1692. Then, this value of θ may be substituted to yield the values of y, y', and M at the various points, as desired. These values may be plotted to obtain values at points other than those used in the computations. The details of this part of the procedure are omitted. Some of the numerical results are as follows:

Item	Description	Quantity
1	Deflection at point a, in inches	40.7
	Bending Stress, in Pounds Per Square Inch, at:	
2	Point b	31,000
3	Point c	29,000
	Percentage of the Total Bending Stress at Point c, for the Moment:	
4	Acting on the undeflected column	62.0
5	Caused by load P, disregarding e	32.5
6	Caused by the weight of the column	5.5
7	Total bending stress (%)	100.0

These numerical values should be considered illustrative rather than accurate, since only four intervals were used in the calculations.

General Remarks.—It is necessary to carry along an unknown in the calculations. In both of the numerical examples selected, this unknown has been θ, the value of y' at a certain point. In some types of problems, however, it is advantageous to choose for the unknown a value of y or of M. Then, with the aid of Eqs. 38*b* and 41, the computations are developed in much the same way as in the illustrative problems of the paper. Finally, a point is reached at which the value of y', y, or M is known, enabling the designer to evaluate the unknown. With the value of the unknown determined, it becomes possible to evaluate y', y, or M at any of the tabulated points.

The computations should be carefully grouped to save time and mental effort. The proper arrangement is determined by the needs of the individual problem.

The method is not self-checking. A procedure that will afford a check on the numerical work and that will at the same time give an indication of the accuracy of the method is to work through the computations first with a small number of intervals and then with a larger number of intervals. A comparison will reveal numerical errors quickly. Also, since the results generally converge very quickly toward exactness as the number of intervals is increased, the difference between the two sets of results gives an approximate indication of the error caused by choosing a finite number of intervals.

The step-by-step method can be applied directly to ordinary beam problems. In that case $M(y) = 0$, and the total moment at any point can be written immediately, thus greatly simplifying the procedure.

5. Actual Stress and Factor of Safety Above the Critical Point

Columns loaded above the critical point, of course, would never be used as structural members; but sometimes they occur in the form of springlike members in various types of machinery.

Only one such column—axially loaded at the ends with negligible end restraint, eccentricity, and irregularity of shape—will be treated. The direct stress will be neglected, since it is generally small compared to the bending stress. With a given column and a known load, δ can be determined immediately by Fig. 3, and the maximum bending stress, $\sigma_b = \frac{P\,\delta\,c}{I}$, can be evaluated readily. The factor of safety, N, may be defined as $\frac{P_y}{P}$, in which P_y is the load producing a bending stress equal to the yield stress. The safety factor may be determined by trial.

The difference between the ideal and the actual column is less marked above, than below, the critical point because, above the critical point, the large deflection is only little affected by minor eccentricities and irregularities of shape whereas, below the critical point, any small eccentricity or irregularity causes the column to depart from its ideally straight position and thus introduces an appreciable bending stress.

6. Summary

The integration of the fundamental differential equation of the simple, ideal column leads to two solutions, one corresponding to a perfectly straight form and one corresponding to a certain deflected shape. Below the critical point, only the first of these solutions exists; consequently, the column must remain straight. Above the critical point both solutions exist, but the first is unstable and the second is stable; consequently, if the column is in the straight condition, the slightest disturbance will suffice to throw it into the deflected position.

With the column straight, the stress consists only of the direct stress P/A. With the column deflected, the stress is chiefly caused by the moment exerted by the applied load at the various points of the column.

In the actual column, end restraint, eccentricity of load, and irregularity of shape often reach appreciable values. If the load is less than the critical, deflections are small, and the elastic curve of the column can be represented accurately by a linear differential equation of the second order with constant coefficients. The solution of this equation is single valued and continuous within the practical range; hence, the shape of the actual column must vary continuously with change of load. In a strict sense, therefore, there is no critical point for an actual column. The sharp mathematical discontinuity that occurs in the critically loaded ideal column is leveled off, to a greater or less extent, by the irregularities of the practical column.

If the irregularities of the actual column are known to be small, the factor of safety at any load may be determined very simply by Fig. 6. Since even small irregularities may have an appreciable effect on the column, it is recommended that the values of Fig. 6 be modified according to judgment, to suit the known or expected irregularities.

The formal solution of the differential equation of the practical column is prohibitively tedious in many cases. Under such conditions, the versatile step-by-step method of numerical integration is suggested.

Above the critical point, the column is not greatly affected by minor irregularities. The numerical solution of columns loaded above the critical point is greatly facilitated by the curves of Fig. 3.

APPENDIX. NOTATION

The following letter symbols, adopted for use in the paper and for the guidance of discussers, conform essentially to American Standard Letter Symbols for Mechanics of Solid Bodies (ASA—Z10.3—1942), prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1942:

A = area of cross section;
C = an arbitrary constant in Eq. 7;
c = distance from the neutral axis of a beam to the extreme fibers;
D = diameter;
E = modulus of elasticity;
e = eccentricity of axial load on a column;
G = an elliptic integral in Eq. 23*b*;
H = an elliptic integral in Eq. 23*b*;
I = rectangular moment of inertia;
i = the length of an interval in Eq. 38*b* = $x_B - x_A$;
J = an elliptic integral in Table 1, defined by Eq. 21;
K = an elliptic integral defined by Eq. 13;
k = a substitution constant = $\sqrt{P/(E\,I)}$ in Eq. 6*c*;
L = the total length of a deflected column;
M = bending moment:

   M_i and M_e = moments resulting from internal effects (M_i) and external effects (M_e); they are denoted "internal moments" and "external moments," respectively;

   $M(x)$ and $M(y)$ = moment (expressed as functions of x and y) at any point of a column, created by externally applied forces;

N = factor of safety; subscripts c and y denote whether the factor of safety is based on P_c or P_y;
n = a number in mathematical progression;
P = an external, concentrated load:

   P' = the value of P that would hold the column in equilibrium;

P_a = allowable load;
P_c = critical value of P;
P_y = value of P that produces a direct stress σ_y;
q = a substitution constant defined by Eq. 12a;
r = least radius of gyration;
s = an indefinite arc length, part of the total length L of a deflected column;
ds = a differential of s;
W = a total weight or load uniformly distributed;
w = a unit, uniformly distributed load; w_w = a unit, uniformly distributed wind load;
x = distances parallel to the longitudinal reference axis of a column;
y = distances transverse to x-distances; lateral deflection of a column at any point:
y' and y'' = first and second derivatives of y with respect to x;
y_e = "exact" values of y (see Table 5);
y_o, y'_o, and y''_o = initial deflection, slope, and approximate curvature at any point;
Z = section modulus;
β = upper limit of the angle ϕ;
Δ = a change in any quantity in a given interval of length i; thus, Δy = a change in the deflection y;
δ = total deflection at the center of a column;
ϵ = maximum deflection of an unloaded column;
θ = slope of a deflected column at a given point;
λ = half the projection of the deflected column on the x-axis (see Fig. 1(a));
ρ = radius of curvature;
σ = normal stress:
σ_b = bending stress at any point;
σ_t = total stress;
σ_y = yield-point stress; and
ϕ = a substitution angle defined by Eq. 10a.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

REPORTS

ORGANIZATION, FINANCING, AND ADMINISTRATION OF SANITARY DISTRICTS

FINAL REPORT OF THE COMMITTEE OF THE SANITARY ENGINEERING DIVISION

The committee submits this as its final report. A comprehensive progress report was submitted at the Annual Meeting in New York, N. Y., in January, 1945.[1] Two years have thus elapsed for discussion of the progress report.

The preamble of the progress report stated that it was intended to stimulate discussion, as follows:[1]

> "The objective of the committee is to formulate the fundamental concepts of the subject assigned to it so as to provide a basis for the preparation of state enabling acts or for the amendment of existing enabling acts regulating the organization, the financing, and the administration of sanitary districts. It is hoped by the committee that there will be an opportunity during 1945 to receive and consider discussions of their progress report * * * and to present a final report in January, 1946 * * *."

During the period of two years since the progress report was submitted, however, there have been only two discussions. For these, the committee is most grateful.

The discussion by C. Maxwell Stanley,[2] M. ASCE, contains many helpful comments. On the whole, it accepts the comments and conclusions of the progress report. With reference to financing the sanitary districts, Mr. Stanley states:[3]

> "The problems of financing a sanitary district are not appreciably different from those encountered in financing any sewerage project * * *.
>
> "It is important that the selection of the method of financing be made only after consideration of prior practices. For instance, if the established practice in the community has been to finance lateral sewers by special assessment, a program to finance additional lateral sewers by general obligation bonds, paid by general taxes, will be unfair * * *."

The committee wishes to caution those undertaking new sanitary district projects to avoid continuing inequitable practices just because they have been

NOTE.—Since the work of this committee has been completed and the committee discharged, this report is not open to discussion.

[1] *Proceedings*, ASCE, February, 1945, p. 177.

[2] *Ibid.*, May, 1945, p. 786.

[3] *Ibid.*, p. 787.

used in the early days before the present-day comprehensive development of sewage disposal projects.

Mr. Stanley's comment[3] that—

> "In considering the over-all financing of sewerage projects, a combination of the three methods involving general taxes, assessments, and rentals would very frequently be advisable."

—is a sound comment. Furthermore, concerning the statement: "* * * the collection of sewage rentals by an addition to water rates is a most suitable procedure * * *"—the committee is of the opinion that this would depend entirely upon the characteristics of the water rates as to whether or not they are really equitable for use in the collection of sewer rentals.

Of particular interest is the comment:[4] "* * * some aspects of sewage projects must be governmental; but water supply, although generally conceded to be proprietary, also concerns public health." It may be that the general conceding of water supply as a proprietary function is wrong and has persisted from the early days when water rates were largely associated with the activities of a private utility having limited powers as to the sources of revenue.

The second discussion was by I. Vernon Werbin,[5] Assoc. M. ASCE, who submitted some interesting references to court proceedings on the question as to whether the providing of sewage disposal service is a governmental or proprietary function. He states:[6]

> "The United States Supreme Court decision in the Brush case, involving a water supply system, is equally applicable to the construction and maintenance of a sanitary system—that the government is 'rather rendering a service than selling a commodity.' "

This designation of a water supply as a governmental, rather than a proprietary, function should be considered in its effect on water rates, many of which seem to be computed on the basis that the water is a commodity and that the service is proprietary.

The progress report of the committee is not easily summarized. As regards the organization of sanitary districts, the committee favored emphasis on local action. Concerning the financing of sanitary districts, the committee called attention to the lack of uniformity in the application of rates to revenue financing, both for water and for sewer works. As regards administration, the committee emphasized that the ability, character, and intelligence of the members of the administrative board are highly important.

Since the progress report was submitted, there have not been many outstanding developments in this particular field; the present status of the problem is approximately as it was two years ago.

The committee recommended that the extent of sanitary districts be determined by a competent engineer, or board of engineers, and that the costs of the necessary surveys and preliminary engineering work should be provided. The committee suggests that this financing be extended to provide not only for

---

[4] *Proceedings*, ASCE, May, 1945, p. 788.

[5] *Ibid.*, June, 1945, p. 929.

[6] *Ibid.*, p. 930.

the necessary engineering investigations to determine the extent of a sanitary district, but also to provide for the preparation of plans and specifications, for the supervision and inspection of construction, and, in fact, for all proper costs until revenues from charges are available. The following amounts are suggested for these costs:

Population	Amount per capita
5,000	$10.00
10,000	7.00
25,000	5.00
50,000	4.00
100,000	3.00
500,000	2.00
1,000,000	1.50

Under the topic "Powers of a District," the committee suggested that these should include the making of rules and regulations for the use of its facilities. Recently, two articles of interest have been published, as follows:

(a) "Control of Sewer Usage at Detroit, Michigan," by Clyde L. Palmer, *Sewage Works Journal*, November, 1946.
(b) "A Report of Procedure for the Handling of Industrial Wastes," California Sewage Works Assn., 1945, W. T. Knowlton, Chairman, 1632 South Van Ness Ave., Los Angeles, Calif.

Some interest has been expressed for the extension of the powers of a sanitary district to include not only the collection and disposal of sewage, but also the collection and disposal of refuse. It would be necessary, in such cases, to provide adequate and equitable funds for both services. This is now under consideration in connection with the preparation of a comprehensive enabling act for the State of Florida for state-wide applicability.

The committee is of the opinion that its suggestions for the organization of sanitary districts are still sound.

The comments of the committee in the progress report on the financing of sanitary districts were, in general, quite tentative and illustrative. The committee asked for further guidance through discussion. It was stated that:[7]

"Even in water rate schedules, which have a much broader basis of experience, there is a marked difference as, for instance, in the number of brackets, the slide or spread between small and large users, and the amount allocated to the public through fire protection service."

There seems to have been little clarification during the past two years. In the absence of extensive discussion, the committee calls attention to two publications of interest. The first is from an article by L. E. Ayres,[8] M. ASCE, as follows:

"There are certain aspects of water rates wherein one finds a wide divergence in existing schedules. Among these are: (1) the range in unit charge as between that made to the average household and to the large

[7] *Proceedings*, ASCE, February, 1945, p. 184.

[8] "A Study of Water Rates in Michigan," by Louis E. Ayres, *Journal*, A.W.W.A., September, 1944, p. 996.

industry; (2) the lack of specific charge, for the most part, for fire protection and public service; (3) the diversity in the application of 'service' and 'minimum' charges; (4) the complexity of many rate schedules as indicated by the number of 'steps' superimposed on a number of meter charges; and (5) the wide range in charges for water sold to 'suburban' areas."

This comment by Mr. Ayres indicates the lack of uniformity in water rate schedules as regards equity and implies a measure of confusion as to whether or not the function of water supply is governmental or proprietary. Recently, a number of municipalities have combined the water supply and sewage disposal services in the matter of financing and administration. This procedure serves to emphasize the need for a sound application of the general principles of equity and function.

Another is from an editorial comment in *The American City*, as follows:[9]

"When one considers the matter, it is perfectly obvious that only an infinitesimal proportion of the water in a municipality is 'consumed.' The rest is 'used.' We find the lawyer, rather than the engineer, driving this point home most convincingly." [See "Sewer Revenue Financing," by R. L. Mitchell, *Daily Bond Buyer*, September 20, 1946.]

* * *

"As Mr. Mitchell has so ably pointed out, the customer merely receives the offering of the water works man, uses it momentarily, and passes it along to the sewage works man."

This comment, without explaining the importance of the term in relation to fundamental equities and functions in charges for sewage disposal service, suggests that the word "user" is more correct than the word "consumer." It is of interest that in all such references in the progress report, the committee used the words "service" and "use" in all cases except one, as illustrated by the following quotations:

(*a*) "Costs of sewage disposal service or of providing for the 'use' of the facilities,"
(*b*) "Costs of providing these several services and 'uses,'"
(*c*) "The remaining operating costs would be a charge on 'users,'"
(*d*) "The slide or spread between small and large 'users,'"
(*e*) "The district would be financed by 'users' and not by property."

The only place where the word "consumer" was used is item 12 in the illustrative schedule of charges, as follows: "Minimum number of consumers." In this case, the word "users" might well have been used.

The progress report of January, 1945, stated that the administrative board should look forward to increasing standards of sanitation. In this regard, reference should be made to descriptions of new standards for waterways receiving sewage and sewage effluents, as follows:

(*a*) *House Document No. 266*, Ohio River Pollution Control, Part II: Report of the United States Public Health Service, August 27, 1943.

[9] *The American City*, November, 1946, p. 9.

(*b*) "Minimum Criteria of Water Quality," presented by the T. V. A. to the Tennessee Stream Pollution Control Board, November 29, 1946.

An interesting standard for the quality has been advanced according to which waterways would be made to conform to raw drinking water standards, rather than to tolerable standards for waterways receiving some sewage pollution.

A bibliography on this subject should include the following recent references:

(*a*) "Sewer Revenue Financing," by Robie L. Mitchell, *Daily Bond Buyer*, September 20, 1946.

(*b*) "Financing through Revenue Bonds," by S. B. Robinson, National Inst. of Municipal Law Officers, 730 Jackson Place, N. W., Washington, D. C.

(*c*) "Financing Sewage Systems and Sewage Treatment," by Francis S. Friel, paper presented at 19th annual meeting F.S.W.A., Toronto, Canada, October 8, 1946. (Abstract appeared in *Water and Sewage*, October, 1946.)

(*d*) "Financing Municipal Projects with Revenue Bonds," by David M. Wood, special I.B.A. convention, *Daily Bond Buyer*, December 2, 1946.

Two interesting court decisions, both confirmed by the higher state courts, should be included for reference, as follows:

(*a*) Gericke versus City of Philadelphia, 44 Atl. (2d) 233 (Pa., October 30, 1945)
Filed March 15, 1945

(*b*) State versus Miami (Fla.), 152 So. 6

In conclusion, the progress report submitted in January, 1945, appears to the committee to be as final and complete a statement on the matter of organizing, financing, and administering sanitary districts as can be well made at the present time.

Respectfully submitted,

George E. Barnes, Elson T. Killam
Harold F. Gray, Arthur D. Weston
Samuel A. Greeley, *Chairman*
Committee on Organization, Financing, and Administration of Sanitary Districts

January 15, 1947

AMI

U

Si
cussic
other
prese
possi
T
three
resul
"in t
origi
print
lost
came
T
valu
A
due
on t
be a
recip
inde
redu
I
acco
this
sion
Hol

posi
Wing

Egy

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

UPLIFT PRESSURE IN AND BENEATH DAMS
A SYMPOSIUM

Discussion

By Serge Leliavsky, and Mohamed Ahmed Selim

Serge Leliavsky Bey.[53,53a]—Those who took part in the interesting discussion of this paper have focused attention on certain important points which otherwise might have remained obscure. In doing so, they have contributed to presenting the subject of this investigation more clearly, thus helping to avoid possible misinterpretation.

The original manuscript from which the paper was condensed was about three times longer than the text finally accepted for publication, with the result that certain points of the solution were left, if not "in the dark," at least "in the shade." This applies particularly to the historical part, which, in the original version, preceded the descriptive account of the tests. Since, in the printed text, this part was deleted, the interpretation of the writer's method lost its historic perspective, and some aspects of the conclusions reached became, therefore, less obvious.

The theoretical "three-dimensional" elastic effect, treated in Mr. Holmes' valuable discussion, was one of these aspects.

According to Mr. Holmes, the failure of the specimens (which is basically due to axial forces) is also affected by the external pressure exerted by the liquid on the lateral surfaces of the porous material. The uplift factor would then be an amount 2μ less than the value computed by the writer (μ being the reciprocal of Poisson's ratio). Assuming that the Brandtzaeg experiments were indeed applicable to the case, the writer's coefficient, $n = 0.85$, would thus be reduced to $n = 0.507$.

Mr. Holmes' point is rather obvious and might easily have been taken into account in the tests, if there had not been specific reasons militating against this procedure. In fact, the objections against introducing the three-dimensional effect in the interpretation of these experiments, as suggested by Mr. Holmes, might have been inferred from the third paragraph following Eq. 16:

Note.—This Symposium was published in December, 1945, *Proceedings*. Discussion on this Symposium has appeared in *Proceedings*, as follows: June, 1946, by W. H. Holmes, W. A. Perkins, and S. P. Wing; and September, 1946, by Gordon V. Richards.

[53] Director, Designing Service, Reservoirs and Nile Barrages, Dept., Ministry of Public Works, Cairo, Egypt.

[53a] Received December 9, 1946.

"Messrs. [P.] Fillunger and [Karl] Terzaghi [M. ASCE] have demonstrated beyond any reasonable doubt that the effect of an interstitial hydraulic pressure upon the resistance of a porous solid is negligible, so long as this pressure is equally distributed over the walls of the pores. Since the existing evidence confirming this statement is conclusive, it follows, in so far as the present tests are concerned, that the ultimate resistance of the test piece is not affected, in all the cases in which $p_o = p_i$. However, the cases in which p_i is smaller than p_o remain to be investigated. When this happens the ring-shaped part of the specimen is subjected to stresses similar to those in a pipe of a boiler. Since these stresses are compressive, z could be smaller for higher values of $p_o - p_i$."

Perhaps this text was not sufficiently clear. Perhaps it would have been better to include the more explicit explanation of Professor Terzaghi's tests in the published version, as it was in the original paper. The fact is that, contrary to the views expressed by Mr. Holmes, unprotected porous specimens, subject to hydraulic pressure, do not behave in the manner described in the standard textbooks on elasticity.

The difference between the results obtained by F. E. Richart, A. Brandtzaeg, Members, ASCE, and R. L. Brown at the University of Illinois, Urbana (cited by Mr. Holmes[44]) with "protected" specimens (provided with thin metal mantels), and the behavior of Professor Terzaghi's own "unprotected" test pieces, was used by Professor Terzaghi as the starting point for his uplift theory. It is not the writer's intention to discuss this theory as such, except to state that there is probably no living man whose factual results could carry greater weight than those of Professor Terzaghi. These experiments, published in 1934,[6] have demonstrated that, when the water is allowed to penetrate into the pores, the behavior of the concrete specimen under axial loads is not affected by the lateral, hydraulically applied, external pressure, referred to by Mr. Holmes.

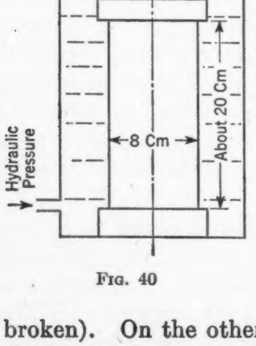


FIG. 40

The Terzaghi test pieces (see Fig. 40) were placed in a vessel, with water under various pressures ranging from zero to 400 kg per sq cm (that is, ten times greater than in the writer's tests), and subjected, at the same time, to an axial load varying from 2.6 tons to 25 tons. The results, at breaking point, are given in Table 15. The values in Cols. 1 are from Professor Terzaghi's experiments with unprotected specimens (that is, similar, in this respect, to the writer's specimens). There is no marked correlation between these data and the pressures, p, appearing in the first column (under which the test pieces were broken). On the other hand, the stresses in Cols. 2 are intended to portray the behavior of concrete specimens provided with thin metal jackets, such as

[44] "A Study of the Failure of Concrete Under Combined Compressive Stresses," by Frank E. Richart, Anton Brandtzaeg, and Rex L. Brown, *Bulletin No. 185*, Univ. of Illinois, Urbana, November, 1928.

[6] "Die Wirksame Flächenporosität des Betons," by Karl Terzaghi, *Oesterreichischer-Ingenieur-und-Architekten-Verein Zeitschrift*, Vol. 86, 1934, Heft 1/2, pp. 1-9.

those of the Illinois tests.[44] These values are scaled from the Terzaghi diagrams, based upon the Illinois tests, in the well-known graphical computation devised by Mohr. The rapid rise of the breaking limit in Cols. 2 is characteristic of the three-dimensional effect referred to by Mr. Holmes. The contrast between Cols. 1 and 2, Table 15, supplies a quantitative negative reply to the question raised by Mr. Holmes.

TABLE 15.—COMPARISON OF COMPRESSIVE BREAKING STRESSES IN KILOGRAMS PER SQUARE CENTIMETER

(Test Pieces Unprotected in Cols. 1 and Protected in Cols. 2)

Pressure[a] p	SERIES I		SERIES II		SERIES III	
	(1)	(2)	(1)	(2)	(1)	(2)
0.......	580	580	532	532	66.3	66.3
100.......	630	860	..	..	67.8	353
200.......	646	1,148	534	1,102	67.0	636
300.......	404	1,431	645	1,390	66.5	922
400.......	604	1,722	454	1,678	67.1	1,210

[a] Water pressure in the test vessel, in kilograms per square centimeter.

The physical explanation of these results will be almost obvious if the analyst will ignore conventional assumptions made in textbooks on elasticity, and will follow the modern views on the subject. In other words, instead of the imaginary, continuous structureless elastic solid contemplated by the mathematicians since the time that Navier, Lamé, Cauchy, and Poisson laid down the bases of classical elasticity, the analyst must visualize the material as an elastic structure, containing both solids and voids (see Fig. 41). This has been the approach of Messrs. Fillunger, Terzaghi, Hoffman, and other modern thinkers. Fig. 1 illustrates the significant aspect of the difference between the two conceptions.

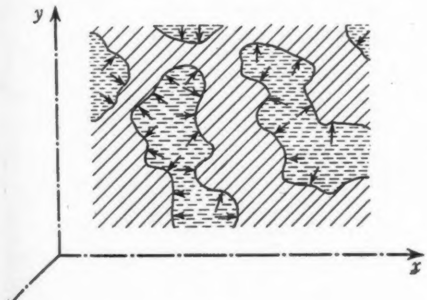


FIG. 41

It is evident from Fig. 41 that the stress conditions in porous elastic solids must depend substantially on whether the pores are empty or full.

Fig. 42(*a*) is reproduced from Professor Terzaghi's paper[6] and represents schematically the conditions of the Illinois tests. In this case the external hydraulic pressure does not differ in its action from a mechanically applied load. The pressure is taken entirely and exclusively by the solid material and, therefore, causes the particles to be pressed against each other. The accumulative effect of their relative displacements results in the elastic deformation of the whole, leading, in its ultimate stage, to the three-dimensional effect exemplified by the failures of test pieces in the Illinois tests (see Cols. 2, Table 15).

The second case, illustrated by Fig. 42(*b*), demonstrates the principles materialized in Professor Terzaghi's experiments. The external pressure is here resisted not only by the particles but, also, by the water contained in the pores. The water thus participates in resisting the externally applied load. The effect on the solid material must therefore be much smaller; but what is

even more important is the fact that, although each solid particle is subjected to a certain uniform compression, there is now no prevailing tendency for one particle to be pressed against the other (except in the proportion of $(1 - n)$ to 1, which is very small indeed). Thus, the chief factor producing strain and

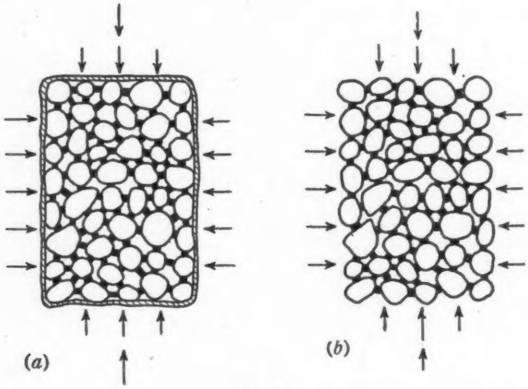


FIG. 42

fracture is negligible in this case; therefore, the pressure can be raised almost indefinitely, without affecting the mechanical resistance of the material (as shown by the Terzaghi tests[6] listed in Cols. 1, Table 15).

The point, thus demonstrated by Professor Terzaghi—namely, the absence of the three-dimensional effect in unprotected porous materials subject to uniform hydraulic pressure—was observed by Fillunger as early as 1913;[7] except that in the latter case, instead of an axial compression, the hydraulic effect upon unprotected concrete specimens, was combined with axial tension. The conclusions were precisely the same, however, as might be seen from the following data, which represent average breaking stresses, for twelve specimens each, of:[14]

Hydrostatic pressure, in atmospheres	Tensile resistance, in kilograms per square centimeter
0	36.1
100	32.4
200	31.6

The significant conclusion is that the resistance of concrete against axial overload is practically the same at the bottom of the sea as in the free air; the fact, therefore, that in the writer's machine the specimen is surrounded by water under a certain pressure, has no elastic effect on the axial resistance of the material tested. There remains only the main axial effect of the interstitial water pressure in the pores of the material, which varies gradually from p_o at the plane of rupture to zero at the top of the specimen, and therefore produces

[7] "Der Auftrieb in Talsperren," by P. Fillunger, *Oesterreichische Wochenscrift für den öffentlichen Baudienst*, Vol. 19, 1913.

[14] "Versuche über Zugfestigkeit bei allseitigem Wasserdruck," by Paul Fillunger, *ibid.*, Vol. 21, Pt. 2, July, 1917, Heft 29, p. 443.

an accumulative vertical action on the walls of the pores, causing the specimen to rupture. Under these circumstances, the writer cannot concur in the idea of introducing a secondary "three-dimensional" effect in the interpretation of his tests, in the manner suggested by Mr. Holmes.

There is another aspect of the same problem, however, which Mr. Holmes does not mention. The writer has shown that, when the pressure, p_o, in the container (see Figs. 7 and 11), and the pressure, p_i, in the inner space of the specimen, are both the same, it is not necessary to take the three-dimensional effect into account. On the other hand, when these pressures are not equal, this effect might possibly become a significant consideration (to the extent of the difference $p_o - p_i$), because the interstitial pressures would then vary, from pore to pore, in the horizontal direction, and the accumulative action of these differences would result in a general horizontal compression of the entire cylinder. This compression, in turn, might possibly affect the axial vertical tensile resistance of the specimen, according to the "three-dimensional" principle referred to by Mr. Holmes.

This explains, more clearly, the object of the specific statistical examination of the results of the tests, described in the last few paragraphs of the paper preceding "Conclusions"—namely, the calculation of the correlation and regression coefficients for Δy (vertical distance from a point to the average line in Fig. 24) and the difference $p_o - p_i$.

Had such a correlation been found to exist it might easily have been taken into account, by a correction based on the regression coefficient; but, as stated in the text, the results of the calculation showed that the correlation in question did not exist. (The calculated values of these coefficients were, respectively, 0.068 and 0.0016.) Even from this rather remote standpoint, therefore, the three-dimensional effect was found to be absent.

This conclusion is also confirmed by the results of Rudeloff's experiments,[13] represented in Fig. 25. In this particular case the specimen (which in other respects was generally similar to Fig. 3) was provided with an impermeable mantel, on the same principle as in the Illinois tests; but, in spite of that, the three-dimensional effect did not exist. The axial tensile breaking load was found to be independent of the hydraulic pressure in the container. The following average values of the tensile resistances of "protected" cylindrical specimens were obtained by Rudeloff:[13]

Hydrostatic pressure, in atmospheres	Tensile resistance, in kilograms per square centimeter
5	7.50
10	8.35
15	7.55
20	7.32

Although these are actual facts which, in themselves, constitute an exhaustive reply to the question under review (in so far as the main uplift problem is concerned), it will nevertheless be relevant to the discussion to explain the

[13] "Versuche über den Porendruck des Wassers im Mauerwerk," by Max Rudeloff and Panzerbieter, *Mitteilungen aus deutschen Königlichen Materialprüfungsamt zu Gross Lichterfelde West*, Berlin, 1912.

apparent contradiction, between the evidence of the Illinois tests on the one hand, and Rudeloff's and the writer's experiments on the other. In fact, the first point that comes to one's attention is that the main axial effect is compression in one case, and tension in the other two cases. The failure is therefore a "shear failure" or a "pseudo shear failure" in the Illinois tests, and a "tension failure" in Rudeloff's and the writer's experiments. Apart from all other considerations, a glance at Fig. 26 will suffice to show that the plane of fracture in the writer's tests is almost exactly perpendicular to the axial breaking force, F, which substantiates the principle of a tension failure, in contradistinction to a rupture by compression in the Illinois tests.

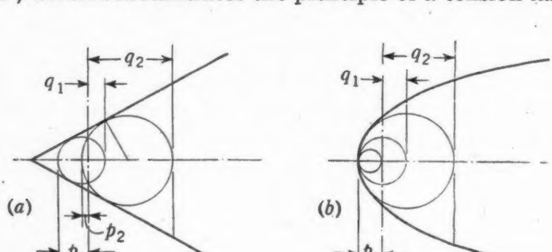


FIG. 43

The difference in the mechanics of the two cases is shown in Fig. 43, of which Fig. 43(*a*) shows the earlier views, with Mohr's critical lines being represented by straight lines. For every compressive breaking stress, q_i, acting at right angles to the main load, there is a different axial tensile limit, p_i. L. Prandtl appears to have been the first to suggest[54] that the critical lines in such diagrams should not be straight, as in Fig. 43(*a*), but curved, as in Fig. 43(*b*). It will then be obvious that the compressive limit, q_i, may vary in rather wide limits without affecting the corresponding tensile breaking stress, p_i. When both stresses are compression, however, the interdependence between the critical limits remains valid, as in the earlier diagrams.

This explains, theoretically, the actually observed difference between the Illinois tests[44] and the writer's experiments (with reference to the difference $p_o - p_i$). Thus, to represent both these experiments in the same chart as is done in Fig. 37 is definitely objectionable. From whichever point the question is considered, there is, consequently, no reason for altering the original conclusions, in order to take into account the "three-dimensional" effect. The writer is indebted to Mr. Holmes, nevertheless, for having raised a most interesting point, and thereby provided an opportunity for an explicit reply.

He is also grateful to Mr. Holmes for having explained in detail the derivation of the formulas he used in finding the empirical straight lines representing the results of his tests for every individual group of specimens. To conserve space, this development, which was based on the least-squares principle, was not given in the writer's paper, for it was believed to belong to elementary mathematics. It is rather unfortunate that in developing these formulas Mr. Holmes changed the signs in basic Eq. 2, which resulted in the second constant of the final equation becoming negative. There is no advantage in this sign convention because the force z is essentially positive, and a negative sign may be misleading.

[54] Report to the Congress of Physicists and Physicians, by Ludwig Prandtl, Dresden, 1908.

In his exceedingly interesting contribution, Mr. Perkins raised a number of important questions. In regard to the convergence of the paths of filtration in the specimen of the paper, as opposed to the would-be parallel lines in a dam, it should be noted that the main flow lines, correlated with the process of failure in the writer's specimen, are vertical. They occur in the material enclosed in the upper lining of the test piece, and are created by, and are a function of, the full difference $p_a - 0$. On the other hand, the convergent streamlines referred to by Mr. Perkins, are only a secondary consideration, depending on the difference $p_o - p_i$, which, as already stated, is negligible as a cause of failure. Apart from that consideration, the lines of seepage are only important in relation to the pressures and it should be understood that the object of the writer's experiments was to determine the effective area on which the interstitial pressures act, and not the pressures themselves, for these pressures have been investigated frequently, and "may now be considered as belonging to the scope of a textbook" (as stated in the second pragraph following Eq. 5). In addition to the references cited in the paper,[17,18,19,20] a comprehensive investigation on the subject has been presented by I. Houk, M. ASCE, and B. A. Bakhmeteff, Hon. M. ASCE.

Regarding the low breaking stress under mechanical tension, the specific series which were marked "S" in Col. 2, Table 2, were meant to constitute conditions occurring in masonry dams (in which indeed the resistance against tension was very small) and were prepared accordingly (see heading, "Procedure and Significance of Tests"). Even in the other series, the general tendency was to approximate current field practice, which does not always coincide with the usual laboratory methods.

The more important point, however, was not in the preparation of the test pieces, but in the testing procedure. As reported in connection with Fig. 19, the effect of the time factor had a large influence upon the observed breaking stress. The dotted points at $p_a = 0$ in Fig. 19, in contradistinction to the point of intersection of the average line with the vertical axis, gave a measure of this effect. Assuming that the compound material was from 60% to 80% as strong as the corresponding laboratory mortar briquette, and that the effect of the time factor was about 2.5 (see Fig. 19), the series marked "C" (concrete) (Col. 2, Table 2) represented a nominal tensile resitance of from 21 to 28 kg per sq cm, which harmonized well with the standard values adopted in Egypt. On the other hand, the general consistency of the observed tensile resistances was confirmed by their obvious quantitative correlation with the various factors on which they must logically depend, such as the method of mixing and the intensity of ramming, the water-cement ratio, etc.

The writer wishes, further, to concur with Mr. Perkins' conclusion, that "* * * a gravity dam with a conservative design of thickness equal to 0.85 H_d

---

[17] "Beanspruchung von Gewichtstaumauern durch das strömende Sickerwasser," by Karl Terzaghi, *Die Bautechnik*, Vol. 12, July 6, 1934, Heft 29, p. 379.

[18] "Security from Under-Seepage Masonry Dams on Earth Foundations," by E. W. Lane, *Transactions*, ASCE, Vol. 100, 1935, p. 1235.

[19] "Uplift Pressure on Dams," by W. Weaver, *Journal of Mathematics and Physics*, M.I.T., Cambridge, Mass., June, 1932, p. 114.

[20] "Permeazioni D'Aqua e loro Effeti nei Muri di Ritenuta," by O. Hoffman, Milan, 1928.

* * * is safe." This, indeed, is an excellent summary of the writer's numerical results.

Turning now to Mr. Wing's discussion, the writer wishes to place on record the pleasure he had in reading and studying this pertinent contribution. The first question raised by Mr. Wing—the apparently low resistance of concrete—has already been answered; but his second point concerning the application of the "least-squares" method, deserves to be explained in greater detail. Eqs. 13 and 14, which are used for finding the constants n and z appearing in the equations of the average lines, are not symmetrical with respect to x and y. The result may, therefore, be affected by the assumed notation; it thus appears to depend on an arbitrary element.

In certain specific cases of engineering practice, this might indeed be of importance, particularly when the recorded observations of the two variables investigated, are not equally precise. As an instance, consider the case of a discharge-level curve, for a section of a large river. In applying the "least-squares" method to this case, the levels should be taken as the independent variable, x, and the curve will then be computed in such a manner that the sum of the squares of the deviations of the individual discharges, y, is a minimum.

Furthermore, when the subtangent angle of the average line ("regression coefficient") differs greatly from 45°, the smaller and larger amplitudes must correspond, respectively, to the y-axis and the x-axis. In the case under consideration, however, the precision of both variables is practically the same, and, also, the subtangent angle of the average line is almost 45°. Consequently, the choice of the variables in this case is immaterial.

The differences in the values of the coefficients in Eqs. 47*a*, 47*b*, and 47*c*, are not substantial numerically, and would not have affected the argument even if they had been physically significant; but, such as they are, they depend on an obvious inconsistency—namely, whichever of the two variables is denoted, respectively, by x or y, the entire computation, from start to finish, must be based on the same notation, whereas Mr. Wing uses the values of z, determined by the writer (for the individual series) on one assumption, but calculates the average curve for ninety-five points, on another assumption. This must naturally affect the result (particularly the second constant).

In regard to Fig. 39, attention is called to the difference between load and stress. For point loads or surface loads, this is indeed elementary and obvious, but in examining the effect of volume loads (such as gravity and uplift) it frequently occurs that one is mistaken for the other, as is the case with the convex curve on the right side of the quoted diagram. This curve is acceptable as representing the uplift pressure, but not as the stress, due to this uplift pressure.

These stresses could agree with the curve assumed by Mr. Wing only if the block of concrete above the section analyzed, were built of an infinite number of concentric rings, having no fixed contact with each other, and capable, therefore, of transmitting no vertical shear. Apart from the inherent physical impossibility of such an extreme assumption, it is also objectionable from the purely theoretical, formal standpoint, because, in that case, the compressive load which is mechanically applied to the upper lining could not be transmitted to the concrete core.

The argument represented by the compression curve appearing in the same figure, is also unacceptable, because overstressing at the edges of the section is the result of re-entrant angles, and the degree of "re-entrancy" is much greater in a standard concrete briquette, considered by E. G. Coker and L. N. G. Filon,[55] than in the writer's specimen. Had this effect indeed been significant, it must have resulted in all the writer's test pieces breaking close to the lower lining, which is by no means the case.

Thirdly, and finally, the stress distribution in Fig. 39, even if it had been true, would not have altered the writer's conclusions, because of the flow of concrete which causes the redistribution of stress intensities when local overstressing occurs. This is the chief point of the modern theory of re-entrant angles in dams, expounded by John H. A. Brahtz.[56]

The next point in Mr. Wing's discussion concerns the time factor, in so far as it affects the conditions of seepage in actual dams. The contention is that, according to laboratory tests with specimens of concrete, the speed of the percolating water is so slow that it would require years for a filtering particle to reach the downstream face of the dam. Had this indeed been the case, the effect of the uplift pressure would have been confined to only a part of the total area of the section—close to the upstream face of the dam.

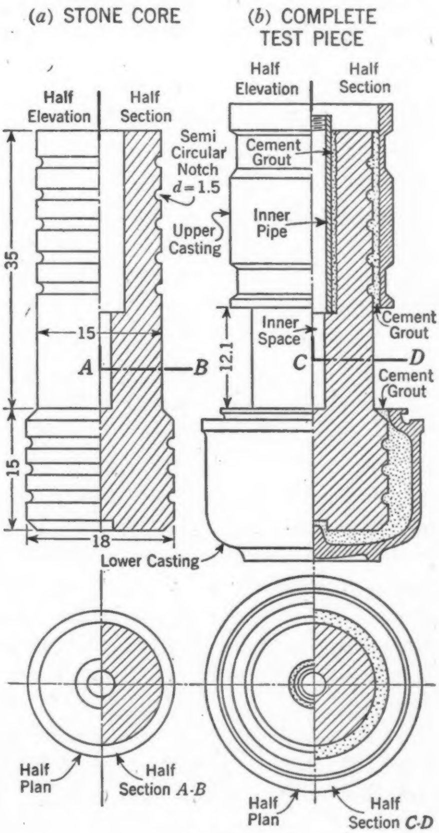


FIG. 44

This contention cannot be accepted for the simple reason that water is actually seen to appear on the downstream face of many dams, in the form of seepage and sweating, in spite of the fact that quite a large percentage must be evaporated. This does not refer to narrow dams only, but even to such conservative designs as the Assuan and Gileppe dams.

It is suggested that Mr. Wing may possibly be overestimating the value of laboratory results, as a practical forecast of the conditions of seepage in nature. In this connection attention is called to a statistical examination of filtering

[55] "A Treatise on Photo-Elasticity," by E. G. Coker and L. N. G. Filon, Univ. Press, Cambridge, England, 1931, pp. 578–583.

[56] "The Stress Function and Photo-Elasticity Applied to Dams," by John H. A. Brahtz, *Transactions*, ASCE, Vol. 101, 1936, p. 1240.

tests with concrete specimens reported by M. Mary[57] in which a wide dispersion of results was observed. What actually occurs in nature seems to be that, adjacent to an almost impermeable section of the dam, there may be a very pervious mass of material, and the designer, therefore, must consider the worst conditions.

In regard to the next point raised by Mr. Wing—the importance of the filtering in the rocky foundations beneath the dam, as compared to seepage in the dam itself—the writer wishes to explain that his method is not confined to compound materials only, such as those used in building dams, but applies also to natural rocks forming the foundations. The specimen is then prepared in the manner shown in Fig. 44, the core being made of the rock to be tested, and grouted in the respective castings. The results obtained in this manner in the writer's machine for Cairo limestone are shown in Fig. 45. These experiments were conducted after the completion of the investigation described in the paper, and, therefore, were not included in the ninety-five points analyzed therein; but they are consistent with the earlier tests.

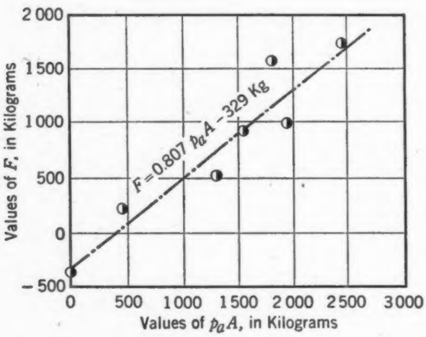


FIG. 45

It was rather interesting to observe that a core prepared from the ignaceous granite of Assuan, and tested in the same manner, proved to be the only material not affected by internal uplift. The writer agrees, nevertheless, with Mr. Wing that, because of the numerous fissures and decomposed seams existing in natural foundations, the uplift must be included in the design even in this case.

Acknowledgment.—For the experiments on which this paper is based, in June, 1946, the writer was awarded the degree of Philosophiae Doctor by Fouad First University, Giza, Cairo, Egypt.

MOHAMED AHMED SELIM,[58] ASSOC. M. ASCE.[58a]—By his discussion, Mr. Richards' has contributed to the practical application of the electric-analogy method in solving problems of two-dimensional and three-dimensional nature. The writer might add that such an application was made at the Laboratory of the Mechanical Engineering Department, University of California at Berkeley, during and after the period of his research; namely, the fall of 1940. Two problems were set up, one concerned with the velocity of air currents and their effect on weather forecasts, and the second with the oil stored in petroleum

[57] "Note on the Accuracy of Permeability Tests," by M. Mary, *Bulletin Periodique No. 7*, Commission Internationale des Grands Barrages, December, 1938.

[58] Lecturer "A," Irrig. Dept., Faculty of Eng., Fouad First Univ., Giza, Egypt; formerly Instructor in Mech. Eng., Univ. of California, Berkeley, Calif.

[58a] Received December 11, 1946.

reservoirs. It is unfortunate that the results of such work have not been presented to the engineering world because of wartime restrictions.

Meanwhile, much work has been done since 1933 by the Ministry of Public Works in Egypt in connection with the lengthening of floor for Assiut Barrage on the Nile. Recent books have shown an increasing confidence in the electric-analogy method of solving the difficult problems of flow through porous media.

The writer is very grateful for Mr. Richards' interest and comments.

Corrections for *Transactions*: In December, 1945, *Proceedings*, on page 1473 and 1474, change, ", Esq." to "Bey." See also errata published in June, 1946, *Proceedings*, on page 898.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

STRENGTH OF THIN STEEL COMPRESSION FLANGES

Discussion

BY GEORGE WINTER

GEORGE WINTER,[35] M. ASCE.[35a]—Since the publication of this paper, the concept of the effective width has been adopted in two structural design specifications: The Specifications for the Design of Light-Gage Steel Structural Members, American Iron and Steel Institute, April, 1946, and the February, 1946, edition of the Specifications for the Design, Fabrication, and Erection of Structural Steel in Buildings, American Institute of Steel Construction. In the latter code, Sections 18*c* and 18*d* make use of an effective width, although in a greatly simplified manner. The writer does not claim any credit for these latter sections. He merely wishes to note the increasing acceptance of this concept in structural design.

The gracious comments of Mr. Llewellyn are deeply appreciated, particularly coming, as they do, from an "old-timer," with such thorough and long-standing interest in this particular field. To Mr. Llewellyn, credit is due for having first proposed the use of an effective width in structural design of light-gage members.[21] Although the values originally suggested by Mr. Llewellyn underwent inevitable correction by subsequent investigation, they were amazingly close to over-all averages, considering the dearth of information on the subject more than a decade ago. Although Mr. Llewellyn differentiated between edges supported by webs, and those stiffened by lips, the writer found that flanges stiffened either way developed the same strength, provided the rigidity of the lip proper was sufficient to furnish full support. This statement holds at least for the range of b/t of the beams of Table 2—that is, for the lipped flanges tested in this investigation.

The only contribution that undertakes to challenge the writer's approach to the problem (merely with regard to flanges stiffened along both edges) is

NOTE.—This paper by George Winter was published in February, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: June, 1946, by Fred T. Llewellyn, and Jacob Karol; October, 1946, by Robert L. Lewis and Dwight F. Gunder; December, 1946, by L. C. Maugh and L. M. Legatski; and January, 1947, by Bruce G. Johnston, and Edward L. Brown and Don S. Wolford.

[35] Associate Prof., Civ. Eng., Cornell Univ., Ithaca, N. Y.

[35a] Received December 12, 1946.

[21] "Light-Gage Flat-Rolled Steel in Housing," by F. T. Llewellyn, A.I.S.C., 1937, p. 29.

that by Messrs. Maugh and Legatski. To evaluate the contentions contained therein, it appears necessary to clarify some obvious misunderstandings, the source of which the writer is unable to trace.

In summarizing their discussion, Messrs. Maugh and Legatski state that "Perhaps the author was led to the erroneous conclusion that the Bryan formula cannot be used to determine the ultimate buckling strength of thin plates * * *." Three lines farther, however, they write:

> "The Bryan load is the load at which the plate element forms into buckling waves, but it is not the ultimate load because a boundary or edge effect remains as an additional element of strength."

This is exactly what the writer maintained in his discussion of fundamentals (paragraph preceding Eqs. 2), where he stated that,

> "The central, more highly distorted, regions of the plate decrease in their resistance, thus throwing more of the total compressive force toward the stiffened edges * * *. This action [failure] occurs at a load higher than the critical—that is, higher than that load at which, theoretically, small deflections start to occur * * *."

These two statements are exactly identical in content, and the writer is at a loss to understand wherein lies his "erroneous conclusion."

In the concluding paragraphs, the statement, "Without accepting the effective width concept as the most logical approach * * *" seems to indicate that Messrs. Maugh and Legatski know of, or have developed, a method which dispenses with that concept. Actually, the approach they propose is likewise based on an effective width, implicitly for stress determinations (see Eq. 23), and explicitly for deflection computations. The writer, again, is at a loss to understand the quoted statement.

From an analytical point of view, contrary to the contributors' contention, Eq. 20, from which the Bryan formula is derived, is no longer applicable, once the critical stress is exceeded. Under such conditions, it is necessary to consider the so-called large deflection theory; that is, Theodor von Kármán's differential equation,[36,37] in which F is the stress function:

$$\frac{\partial^4 \eta}{\partial x^4} + 2\,\frac{\partial^4 \eta}{\partial x^2\,\partial y^2} + \frac{\partial^4 \eta}{\partial y^4} = \frac{t}{D}\left(\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 \eta}{\partial y^2} - 2\,\frac{\partial^2 F}{\partial x\,\partial y}\frac{\partial^2 \eta}{\partial x\,\partial y}\right)\ldots(29)$$

which depends not only on the external compression force, but also on the deformations. It is the extreme complexity of Eq. 29 which is the reason that a solution, so far, has been obtained for circular plates, only. In other words, once the critical stress is exceeded, the plate assumes a new state of equilibrium, which is governed by factors greatly different from those which determine the Bryan load. Eq. 20 does not hold for this state of equilibrium, because, in this state, the stress s is not uniaxial and constant throughout the flange.

An analytical investigation[38] of the stresses above the critical, by means of

[36] "Encyklopaedie der Mathematischen Wissenschaften," Vol. IV/4, 1910, p. 349.

[37] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York and London, 1st Ed., 1936, p. 323.

[38] *Ibid.*, pp. 390–395.

approximate strain energy methods, not only shows the stress distribution to be exactly of the type of Fig. 12; it also indicates that, contrary to the assumption made by Messrs. Maugh and Legatski, the stress in the center is not constant and equal to the Bryan stress. For very wide and thin plates (as Professor Timoshenko has shown[39]), the center strip of an edge-compressed plate is subject to tension, rather than to the Bryan compression stress. The same result, in a rigorous manner, was obtained by Messrs. Friedrichs and Stoker for a circular plate.[9]

To summarize: The writer did not maintain, as is stated by Messrs. Maugh and Legatski, that the center strip is free of stress. This is evident from Fig. 12, whose validity is borne out by the contributors' own strain measurements (for which the writer is grateful indeed as collateral evidence). However, he does maintain that the stress at the center strip is not constant and equal to the Bryan stress, which is again confirmed by the contributors' own stress measurements (Fig. 18(*a*)). These show a decrease of the center stress by about 40% of its maximum value with increasing loads, even at a b/t-ratio as moderate as 98.

In view of the theoretical complexity of the problem, the writer pursued a frankly semi-empirical method. Messrs. Maugh and Legatski propose instead a semi-analytical approach, based on a number of arbitrary assumptions, one of which is discussed herein (that is, center-strip stress equals Bryan stress). Another such assumption is that of taking the proportional limit as five eighths of the yield point, which was apparently necessary to make the equations fit the Schuman and Back results. Even for steels, the so-called "proportional limit," is a completely fictitious quantity, whose value depends primarily on the investigator and his instrumentation.[40] In sharply yielding steels, in addition, the error introduced by equating the proportional limit to the yield point is almost always negligible as compared with inevitable errors, due to other unknown but important factors (edge restraint, initial distortion, etc.).

In their method, Messrs. Maugh and Legatski like the writer, introduce an effective width, b_e, relating it, however, only to the part of the stress in excess of the supposedly constant critical stress. Their final answer is given in terms of a mean stress, s_u, related to the entire unreduced flange area.

For designing members in uniform compression, it may be irrelevant whether the actual stress distribution, Fig. 12, is replaced by a uniform mean stress, or by the writer's equivalent width, related to the maximum stress.

For designing members in flexure or eccentric compression, however, it is necessary to determine the actual maximum edge strain; because it is that strain which governs the location of the neutral axis, without which stresses cannot be computed. The writer's method allows this determination, the contributors' does not. The latter, therefore, would determine flange stresses by the ordinary flexure formula, using the centroidal axis. How far the actual neutral axis, as determined by strain measurements, can deviate from the

[39] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York and London, 1st Ed., 1936, p. 395, Fig. 207.

[9] "Buckling of Circular Plate Beyond the Critical Thrust," by K. O. Friedrichs and J. J. Stoker, *Journal of Applied Mechanics*, March, 1942, p. A-7.

[40] "Stress, Strain and Structural Damage," by H. F. Moore, *Bulletin No. 10*, Univ. of Illinois, Urbana, Vol. 37, 1939, pp. 17–18.

centroid is shown in Fig. 22. The results there given were obtained on two beams of practically identical over-all dimension, but with different sheet thicknesses. The error that would result in using the centroidal axis instead of the actual neutral axis for design computations is evident from the figure.

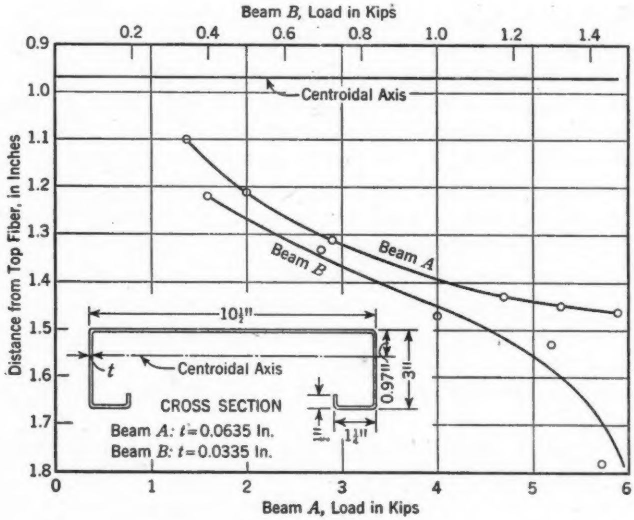


FIG. 22

This situation is implicitly conceded by Messrs. Maugh and Legatski in their statement that, for deflection computations, an equivalent width must be used. For reasons not indicated this width, for deflection computations, is different from the width used implicitly by the contributors for stress computations, as given in Eq. 23. Effective widths, locations of neutral axes, and deflections are interrelated purely geometrically, inasmuch as they are all functions of the magnitudes of the extreme fiber strains. If an effective width approach is necessary for determining deflections (that is, neutral axes), it is physically contradictory to maintain, on the other hand, that stresses can be computed from unreduced widths. This is tantamount to stating that a beam has two neutral axes—one, referred to the unreduced section, governing stresses, and another, referred to the reduced section, governing deflections. In the writer's proposed method, the same approach applies to both stress and deflection determinations.

Parenthetically, the contributors' method (computing deflections by (*a*) using an unreduced moment of inertia to find part of the deflection, (*b*) computing a critical stress, and (*c*) determining a second, reduced moment of inertia for finding the additional deflection) is certainly more cumbersome than that proposed in the paper. It results, qualitatively, in the same behavior as that found by the writer; that is, the effective moment of inertia (in the contributors' case, the weighted mean between the full and the reduced one) decreases with increasing b/t and increasing stress. It is rather doubtful, how-

ever, whether the arbitrary effective width of $40\,t$ for the reduced moment of inertia, given without justification or empirical verification, leads to reasonably accurate results. For the wider flanges (as used in many commercial decks with high b/t) where the critical stress is very low, the use of $40\,t$ will be found to lead to erroneous results on the conservative (uneconomical) side in determining deflections at design loads—that is, at stresses of from 10 to 20 kips per sq in. (For such conditions effective widths in this investigation were found to be of the order of $60\,t$ to $80\,t$ as compared with the contributors' $40\,t$.)

With regard to the contributors' criticism of test methods, the writer will agree to the extent only that (1) strain measurements on light-gage sections are plagued by various disturbing influences not present in more solid members; and (2) that it is always possible to think of other methods of instrumentation that could have been used. With regard to the latter, however, the time element is an important consideration. In this investigation, it was necessary to cover a very wide range of dimensions, which can be done on a large number of specimens, only. The investigator is limited, therefore, to such measurements as will furnish the most pertinent data. This, it is believed, was achieved by the methods selected. The collateral, confirmatory evidence provided by Messrs. Maugh and Legatski, by different test methods, is therefore doubly welcome.

The contributors' statement that twist and lateral deflection may have distorted the measurements, can be discounted, except for possible microscopic effects. Beams of the shape of Fig. 3 have no tendency toward lateral deflection, particularly if loaded through rollers with axes perpendicular to the axis of the specimen. In fact, to deflect laterally, they would have to overcome the contact friction at the load points. Not only can it be shown by computation from known friction coefficients that this is impossible, but any such sliding motion would immediately manifest itself by a scraping sound, as the writer observed frequently in tests of an entirely different nature. Twist, on the other hand, was prevented by a loading arrangement, which forced the beam into parallel vertical displacement. It is evident that effects of the magnitude of those of Fig. 22, consistently obtained on a great number of specimens, cannot be ascribed to details of instrumentation. Finally, the problem of the linearity of strain distribution in such specimens, even if it were open to question, appears to be irrelevant. The location of the neutral axis, as used in the proposed methods, is merely a means to determine, in design computations, the relative magnitude of the top and bottom strains. Since these strains were measured directly, even a curvilinear transition would not affect the results of such design computations. In addition, however, no reason for a curvilinear strain distribution is given by the contributors; nor is any apparent to the writer, particularly since strain measurements were made in the center part of the quarter-point loaded beams, that is, in a region of pure bending.

Two of the contributions (Messrs. Lewis and Gunder's, and Mr. Karol's) do not question the findings of this investigation, but concern themselves mainly with the mathematical form of Eqs. 6 and 7. Formulas such as these are developed with due consideration to simplicity of application. In this particular case, it was imperative, for simplification of design, to delimit a range of b/t, for

which the full width can be used (up to $b/t = 25$). Although, theoretically, the establishment of such a definite limit is questionable, it is completely justified practically. This is why seemingly elaborate mathematical expressions had to be chosen, which result in $b_e = b$ for this limit, independently of the stress. The mathematical complexity of the formulas is relatively irrelevant, since designers will work from graphs such as Fig. 7, rather than from equations. In addition, Mr. Karol's Eq. 13, which involves a hyperbolic function, for that reason, will not be found very convenient by most designers.

Mr. Karol's cautioning remark regarding the use of the writer's chart (Fig. 7) for materials other than steel is to the point, and should be considered in such applications.

The writer gladly accepts Messrs. Lewis and Gunder's justified criticism of the statistical method used in this paper. It is quite true that more sensitive devices, such as standard deviations, should be used in determining the accuracy of fit of empirical formulas.

The writer appreciates the painstaking work of Messrs. Lewis and Gunder in replotting Fig. 2. He does not believe that Eq. 18 should be used practically, despite its simplicity. On the one hand, as the contributors state, goodness of fit of Eq. 18 within the tested range is not as satisfactory as that of Eqs. 6 and 7. Also, tests conducted since the publication of the paper showed Eq. 6 to be applicable satisfactorily for values of b/t up to 400 and more (see contribution by Messrs. Brown and Wolford). In this range of high b/t, Eq. 18 would err on the conservative side up to about 25%. On the other hand, Eq. 19 is identical in structure with Eq. 6, except for minor differences in the constants, and, as stated by the contributors, is not significantly better. The writer is grateful for the fact, established by Messrs. Lewis and Gunder, that, despite his somewhat cursory methods of curve fitting, he apparently managed to arrive at expressions which are as accurate as those developed by more sensitive statistical means.

Professor Johnston furnishes interesting additional material on flanges stiffened along one edge, by reference to his and Professor Cheney's earlier tests on wide flange sections.

The failure criterion used by Messrs. Johnston and Cheney differs from the writer's in that they noted merely the ultimate loads, whereas the writer considered also the stress at which local wrinkling was first noted. Therefore, Professor Johnston's results should be compared with the values of the ultimate stress, s_u in Table 4. In the range of b/t, common to both investigations (that is, up to about 21), it is seen from Table 4 that the writer obtained ultimate stresses of 0.8 to 0.9 of the yield point, with very few exceptions. This is in general agreement with Professor Johnston's findings of a 90% yield-point strength. The somewhat lower values obtained by the writer on thin gage specimens are probably due (*a*) to greater edge restraint in the H-sections, as noted by Professor Johnston, and (*b*) to the better forming accuracy of the milled H-beam flanges, as compared with cold formed sheet steel elements. The writer's colleague, Professor Cheney, volunteered the information that in many of the tests cited by Professor Johnston, development of slight waves and

kinks was noticed at loads below the ultimate, which is in agreement with the findings of this investigation.

The writer wishes to thank the American Rolling Mill Company for releasing for publication the test data given in Messrs. Brown and Wolford's contribution. These data furnish important, supplementary evidence, since the b/t-ratios of these specimens, 242 to 429, are far beyond the range of those discussed in the paper. The test results given in Table 6, as well as conclusions by Messrs. Brown and Wolford, confirming the validity of the proposed method in this high range of b/t, speak for themselves. It is seen that the determinations made on the basis of Eq. 6, if compared with test results, are slightly on the conservative side, both for deflection and for strength. The deviations are small and within the range of discrepancy observed in most structural testing, but they point to one factor that should be emphasized in closing.

The behavior of thin compression flanges is naturally influenced by the amount of rotational edge fixity provided by adjoining elements, such as the webs in Fig. 3. A reasonably simple design method cannot be expected to take explicit account of this involved factor, particularly since, in the great number of tests, its numerical effect was found to be small. The amount of restraint provided in the very wide flanges by the shallow webs of Messrs. Brown and Wolford's specimens is rather large, as compared with a number of other beams in this investigation, with a consequent strengthening effect reflected in the test results.

The proposed methods, therefore, merely claim to represent average conditions of restraint. However, it is important to note that, despite a great variety of conditions of support and restraint obtained in these tests, deviations computed from observed results were within narrow limits, acceptable in practical work.

Correction for *Transactions:* In February, 1946, *Proceedings*, Table 4, on page 219, for type I-B-4, $s_u = 40{,}300$.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852

DISCUSSIONS

LANDSLIDE INVESTIGATION AND CORRECTION

Discussion

BY HYDE FORBES

HYDE FORBES,[19] M. ASCE.[19a]—The discussions have presented material and considerations which will point the way for the application of methods of landslide investigation and correction developed in one region to conditions obtaining in other regions—which is the primary purpose of the preparation of the paper. Professor Legget states that the writer placed emphasis on the development of instability of San Francisco (Calif.) areas due to the presence of serpentine, yet states that the rock formations are not unique. Large areas in San Francisco are underlain with serpentine and are stable, providing sound foundations. The writer had in mind the fact that all rock disintegrates and decomposes under certain conditions, the ultimate product of that decay being largely hydrated material—clay.

Rock containing a high percentage of magnesian minerals has an affinity for water which gives it a tendency to hydrate. This characteristic makes the peridotite of the San Francisco area a ready subject for metamorphism into serpentine, and its surface is subject to comparatively rapid geochemical change when in contact with water. Other common basic igneous rocks such pyroxenite, diabase (Market and Glendale slide), and some gabbros have exhibited metamorphism and extensive decay; sedimentary formations (Crolona Heights, Bernal Avenue cut, and approach to Broadway Tunnel) develop a blanket of clay that becomes unstable with the absorption of moisture. The adaptation of the methods described to areas of unstable clay accumulation is believed possible regardless of the origin of that clay. Test boring and sampling has shown clay to resist penetration and become consolidated to a greater extent with depth below its surface. Clay becomes stabilized by the drainage of overlying or underlying materials, or by the substantial reduction of the time the clay surfaces are exposed to water, together with provisions for drying the clay.

NOTE.—This paper by Hyde Forbes was published in February, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: May, 1946, by T. W. Lambe; September, 1946, by Jacob Feld, George S. Harman, and Robert F. Legget; November, 1946, by Rush T. Sill, Earl M. Buckingham, and Alfred V. Bowhay; and December, 1946, by D. P. Barnes.

[19] Cons. Engr. and Geologist, San Francisco, Calif.

[19a] Received December 5, 1946.

It is of value to know that the horizontal borehole method of installing drainpipe, described by Mr. Buckingham, has been effective in arresting slides in Oakland, Calif. The method is particularly adapted to tap water under "channel "conditions and the ability to locate such channels appears to be the important factor. The writer used that method successfully to tap free water in fairly coarse granular materials in a gully under a fill. The base of the fill had become saturated and the fill was sloughing. In a similar situation, in fine dune sand, the perforations of the drainpipe soon became clogged and the diffused ground water continued to accumulate. The limitations of the percolation or drainage area presented by a small diameter pipe where ground water is diffused through the material, and the possibility of clogging when installed through plastic clay, made the method not applicable to the conditions found to exist in the clay involved in movement at most slides examined. The clay could only be stabilized through drying out, as described as a function of the shafts installed (O'Shaughnessy Boulevard slide).

Mr. Bowhay discusses the "maintenance" required in the earlier work done by the City of San Francisco in connection with landslide correction and stabilization. The object of the later work described by the writer was to avoid such maintenance and to provide permanent stabilization, which seems to have been accomplished. The point is well made by Mr. Bowhay that expedient or temporary measures are never satisfactory, and they are subject to criticism and complaint by taxpayers when eyesores or hazards continue to exist. The design of the slide correction works on Parker Avenue followed the recommendations of the consultants who had been investigating and observing conditions for a period of about one year but were not retained during the design and construction operations. Subsequently, the procedure of advisory and supervisory employment outlined by the writer (see heading, "Continuing Investigation During and After Construction") was adopted by the engineering department. In regard to the future improvement of Parker Avenue, it is believed the necessary extensions can be made at far less cost per unit of length than that of the original work. Mr. Feld emphasizes the advantages of test pits over borings in determining subsurface conditions. It is true that there are advantages; and the continued observations of those conditions in shafts excavated as part of the slide correction works described has revealed necessary modification of original plans based on preliminary investigation. This has also resulted in savings in cost below original estimates.

The lack of correlation of drain discharge from various slide areas with the San Francisco rainfall record noted by Professor Legget was, in part, attributed (see heading, "Preventive and Precautionary Measures") to the increase in effective radius of the works with time, as natural subsurface channels are developed to carry water toward them. In certain areas a decrease in the discharge has been noted as the materials are dewatered. Becoming more compact or consolidated, the surfaces tend to shed the water that falls upon them rather than to absorb it. In further answer to Professor Legget's discussion, the writer's reference to the development of certain laws and values through the use of dry silica sand applied to the treatise by William Cain[20] pre-

[20] "Earth Pressure, Retaining Walls and Bins," by William Cain, John Wiley & Sons, Inc., New York, N. Y., 1916.

sented in 1916, using the theories of Rankin and Coulomb in relation to the factors of cohesion and internal friction.

Values had been determined experimentally, for these factors, in connection with clean sand. The writer has been impressed by the number of abstract considerations presented since which were adaptations of that treatment involving those two factors. Values for cohesion and internal friction had been assigned arbitrarily within the range of those determined by Mr. Cain, or through experiments made on samples, the approach being in part theoretical, in part experimental, or, frequently, a combination of the two. The criticism has been made that geologists lack the training in mathematics and mechanics necessary to comprehend such treatment fully, but empirical methods of stability analysis as applied to materials in natural occurrence have been questioned by geologists because nature does not act in an empiric manner. The factors of cohesion and friction are not static, being affected by slight changes in moisture content and, principally, by the continued geochemical change occurring within a body of material.

Mr. Barnes recognizes this as he states, "* * * the secondary factors such as reduction in cohesion or in coefficient of friction and swelling may outweigh abstract conclusions." The colloidal silt and clay particles present or developed in accumulations of sand or other natural material tend to lessen internal friction and, being favorable to absorption of moisture, may render it more cohesive or unstable dependent upon the amount and the mineral character of such material and its absorptive capacity. Water percolating through sand separates the grains, thus reducing friction; and its presence tends to facilitate a change in form from crystalline or fragmental to the amorphous or hydrated form. Mr. Feld enumerates, as one of the internal adjustments causing landslide displacement, "(*c*) a modification of the soil constituents causing a decrease in physical resistance characteristics." Such modification is due to geochemical change and the point the writer wished to make was that the change is continuous.

The admonition for caution in the use of mathematical solution for stability problems, commented upon by Professor Legget, resulted from the writer's experience. Called upon to investigate landslide and other conditions of instability and present findings in court in connection with claims for damages, the writer has been confronted with mathematical computations, asserted to be supported by texts or other publications and to prove the effect of certain acts. In one case a slide occurred because of geological conditions affected by a long period of unusual rainfall. The court took judicial notice that the same conditions generated numerous slides in the same region during the same period. Quoting from the decision of the judge:

> "The expert testimony of Mr. Forbes, together with his factual observations of the locality in and around the slide, afford an adequate and scientific explanation of the cause of the slide, with which the court is in accord and to which it gives credit."

In another case involving a claim for damages in a condemnation suit the decision of another judge stated:

"I am not at all in accord with defendant's witnesses. I do not believe the big slide * * * was due to excavations of the plaintiff. I believe it was due to natural causes, including the steep slope of the ground, the character of the soils, the rains, and the drainage. * * * the description [by the witness] of the geological conditions of the areas under examination, and reasons for the slips and slides there found, seemed to me to be essentially sound."

Mr. Buckingham points out that landslides are the natural "processes by which topographic forms are sculptured," and the writer has found the cause to lie more often in geologic reasoning based on investigation, which was the basis of the decisions cited, than in abstract consideration of the effect of near-by work. One of the San Francisco slides was treated theoretically, values of unknowns being determined experimentally on samples of the material taken from the slide, and a mathematical solution of the factors and evaluation of the acts generating the slide was published in a bulletin of a public agency. The writer was later employed to investigate the slide and recommend methods of correction. That investigation revealed the slide to have been generated, principally, through ground-water accumulation and uplift, which were not considered in the published treatment. The writer appreciates the value of abstract treatment of stabilization problems, as well as the limitations; he studies the problems and frequently uses them as a tool in analysis and, much less frequently, in making graphic presentations to illustrate the principles involved and the reasoning followed in the design of works for which he recommends the expenditure of money. Also, in order to obtain some idea of the safety factor the works provide, the writer adopts the abstract treatment, with the revision of values assigned to unknown factors as he "believes" they may become changed through the installation of works. He does not use it as a substitute for adequate geological investigation, however. Mr. Buckingham's comment in this regard finds the writer in full agreement. The writer's comment has been previously presented.[21] The uncertainty in applying the results of laboratory determinations of values to field conditions lies in the fact that it is never true that a body of natural material will exhibit uniformity of physical properties consistent with tested samples; nor will those properties remain constant for any considerable time.

Mr. Lambe presents an interesting treatment in relation to the effect of submergence which, in some respects, is confirmed by Mr. Barnes' analysis. In that connection, it may be of interest to note that a subaqueous landslide is a common occurrence. The foundation for the south pier of the Golden Gate Bridge was originally planned as a caisson to be sunk and landed at a predetermined depth below rock surface. Excavation was carried on within a prescribed area into which the caisson was to fit and the caisson was built. When the matter was referred to the writer by the contractor's engineer, it was estimated that about twice as much material had been excavated from the subaqueous site as could be accounted for by the volume of material within the boundaries of the excavated area. Divers were sent down and found that landslides had occurred from all sides. The caisson was abandoned, towed

[21] *Transactions*, ASCE, Vol. 110, 1945, pp. 340–341.

out to sea, and sunk, and a new foundation was designed to fit the conditions resulting from the sliding walls of the excavation. The problem of possible underwater slide shearing structures was presented in connection with the intake towers at the end of the new (1945) adits for the San Andreas Reservoir (Crystal Springs Lakes) of the San Francisco Water Department. Many alluvial fans and detrital accumulations on the floor of the reservoir were found to have suffered landslides with saturation on submergence, and these old slides were revealed when the reservoir water level was lowered. The slide surfaces were examined and the slopes found to be as flat as 1 on 2½ to 1 on 3. Similar material surrounding the towers was graded to those slopes, therefore, before the water level was raised again.

Slide analysis on the basis of displaced volumes described by Mr. Buckingham has its value as well as its sources of error. The moving mass most generally breaks up to a number of units horizontally and seldom is the movement uniform in rate from top to bottom. A large slide along Camino del Mar, the scenic drive following the Golden Gate in San Francisco, was investigated in 1946. Certain lower areas of the slide exhibited movement at rates in excess of those taking place in adjoining areas, due to a higher moisture content of the clay, and so that striated scarps were left as banks to each "glacier-like stream" of clay. During the winter 2-in. pipes were set in boreholes, and carried through the displaced fill, underlying sand, and clay into bedrock in some instances, and in others ending in the clay above bedrock. By late summer the pipes reaching into rock had been sheared at rock line and those ending in clay were dragged so their projecting ends pointed up the slope; the clay in which they were landed moved at a faster rate than the overlying sand and fill. The maximum thickness of moving ground was 125 ft, composed of 35 ft of fill, 45 ft of moist sand, 10 ft of saturated sand, and 45 ft of clay. The latter was the plastic product of the decomposition of the bedrock which had accumulated to that thickness in the bottom of a bedrock gully through movement from the sides and down the slope of the gully. Such differences in rates of movement are important in the location, depth, and type of stabilization works selected. The effect of movement on the many observation pipes set in the Parker Avenue boreholes showed that the moving mass was separated vertically in horizons of varying character of material and varying groundwater conditions, each horizon with a distinct character and rate of movement; all horizons had to be drained and stabilized down to bedrock. This has been usual and the stabilization works have been carried into the bedrock and drainage pipes have been set in that rock in most of the installations. That is an added reason why the horizontal borehole treatment does not apply to the slides discussed, although successfully applied to the Oakland slides described by Mr. Buckingham.

Mr. Feld adds greatly to the value of the paper in citing the application of drainage methods to conditions of instability developing within a body of material, although exterior displacement, or landslide, does not result. Mr. Feld agrees with the writer as to the necessity of obtaining a complete picture of subsurface conditions and it is evident that he has a good understanding of geology and geological phenomena; but he finds it difficult to

agree with the writer that boring records and samples are of little value if not collected by an experienced geologist. It is probable that the writer did not make himself clear on the point.

Since about 1917 contractors and engineers have submitted for interpretation, driller's logs, in some instances accompanied by samples and frequently attached to specifications. Such material was of no value, except when taken from localities familiar to the writer through previous work or in connection with structural requirements which presented simple problems, as it was impossible to produce an interpretation having the reliability necessary when a contractor is to risk his money in making a bid, or upon which an engineer must depend in design; and the services sought had to be refused. In years past, the City of San Francisco has let many contracts for boring which produced only driller's logs and bottled samples and for which there was left no record of interpretation. The writer has found, and wishes to convey in his statement, that a geologist experienced in problems presented, which require further subsurface information, after making a careful study of the surface conditions and history of the locality in question and its vicinity, can direct the location of a minimum number of borings or test pits to produce the maximum working data. During the progress of drilling and sampling, he can observe mechanical actions which are informative; he can determine ground water and soil conditions at sufficiently close intervals so that he can project ahead and tell drillers and inspectors what to watch for and note during his absence; and he can make current geological interpretation while matters are fresh in mind. The saving in cost of exploration in such instances has more than covered the cost of services rendered and the results obtained have been used with confidence which proved warranted. Mr. Harman mentions the value of geological investigation and direction of work in accordance with the results of such investigation; but an amplification of his experience with landslide correction and stabilization work on city-owned land conducted as "relief projects" would have explained his views.

Mr. Buckingham presents the reasoning he follows in the analysis of landslides through surface topographic features. As a rule much can be derived from such observation, but the exceptions are important, and occur frequently enough to make subsurface observation a requisite. No evidence was found that sliding had occurred on the slope at St. Mary's Playground before the the fill was placed. The fill compressed the soils through which water previously passed freely down the slope; pressure was developed and the fill slid out of place. At Parker Avenue the uniform sand slope of Lone Mountain gave no evidence of the underlying bedrock gully. Water was found in disintegrated bedrock underlying residual clay in that gully, and in sand beds between clays which had been developed when those sands were surfaces. The flow net, as modified by the escape of water from the disrupted mass during a period of more than a year following the slide, is shown by contour lines in Fig. 21. The variation of water level in the holes to bedrock (circled) from those not registering the deeper water at that time is noted in numerals.

Mr. Lambe designates the writer's consideration of the ground-water hydrology involved in the generation of landslides as being described "in gen-

eral terms which, to the engineer, do not present a clear picture of the true nature of the effects." This view is undoubtedly warranted, as a consideration of the subject, even in connection with but one of the slides, proved to add too greatly to the length of the paper. The continued observation of the water levels in pipes set in preliminary boreholes has provided very necessary data

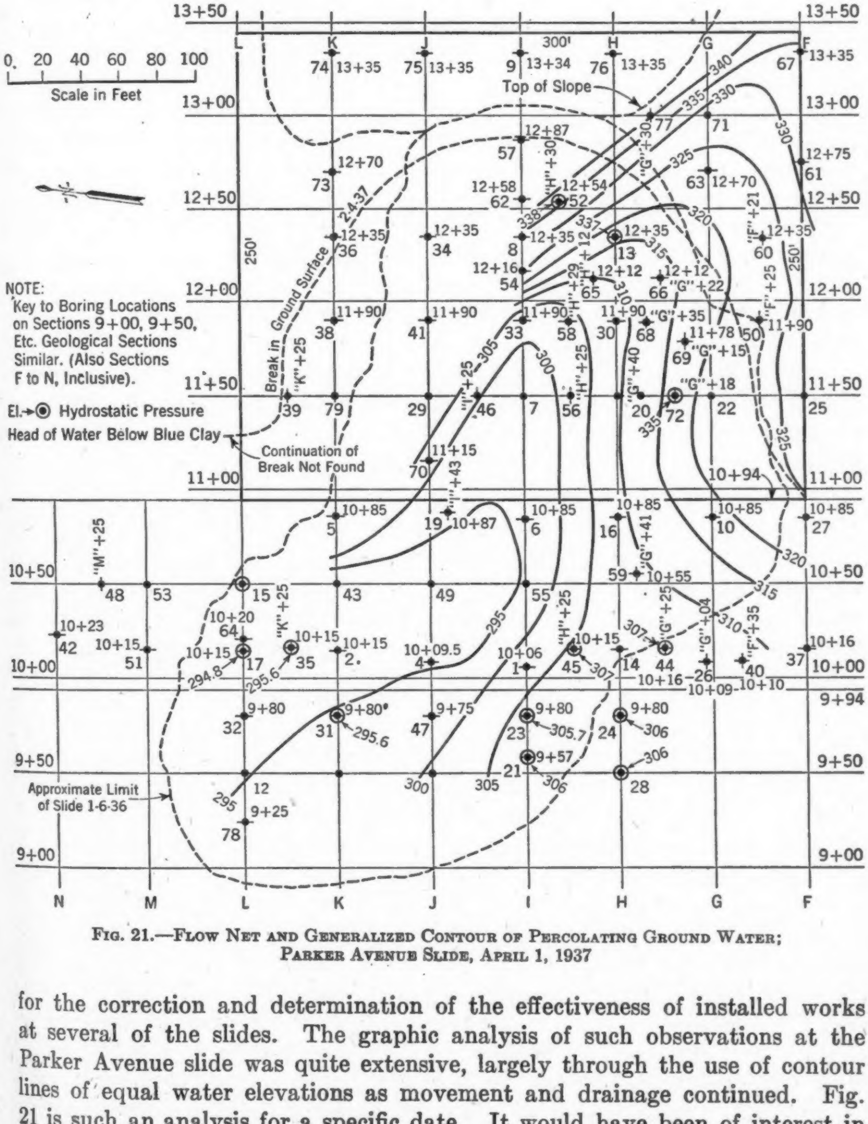


FIG. 21.—FLOW NET AND GENERALIZED CONTOUR OF PERCOLATING GROUND WATER; PARKER AVENUE SLIDE, APRIL 1, 1937

for the correction and determination of the effectiveness of installed works at several of the slides. The graphic analysis of such observations at the Parker Avenue slide was quite extensive, largely through the use of contour lines of equal water elevations as movement and drainage continued. Fig. 21 is such an analysis for a specific date. It would have been of interest in Mr. Harman's discussion of the St. Mary's Playground slide had he presented his analysis of the water levels measured in pipes set in many borings around

the slide and covering several years. A water table could be defined for the area; but many inconsistencies were found in individual measurements which were not explained until the boring for the correction work was under way.

Mr. Sill describes in detail the geological conditions resulting from the weathering of granitic rock, and the slippage of masses of such rock lying between a steeply dipping fault plane and the excavated face of a spillway. It is of interest to note that movement was arrested by preventing the entry of surface water. It has been the writer's experience that finely divided fault gouge acts as a lubricant as well as a plane of parting, and that dry masses will part from the main body and slide along the slanting footwall of a fault zone if excavation leaves them unsupported. A similar experience was had with micaceous schist. The State Highway Department blasted a cut in that rock over a Western Pacific Railroad tunnel near Keddie, Calif. The blasting caused the mass to part along planes of schistocity dipping about 70° from the horizontal toward the cut and tunnel which, although dry (summer of 1931), allowed segments to slide along the slick planes into the highway cut. Other segments created a thrust against the tunnel lining at a depth of about 60 ft below the bottom of the cut, buckling it.

The writer is gratified by the interest shown in the subject matter of the paper, and appreciates the value of the comment and material produced by discussion in its application to the general problem of landslide investigation and correction.

Correction for *Transactions:* In November, 1946, *Proceedings,* on page 1234, in Fig. 16(*a*) change "Sept. 18, 1941" and "Feb. 18, 1945" to "No. 18; Sept., 1941" and "No. 18; Feb., 1945," respectively. Add a point *K* near the right-hand edge of Fig. 16(*a*). In Fig. 17, insert a circle in the legend to identify "Manholes" and adjust the channel arrows in Fig. 17 slightly to conform to the actual courses; and, on page 1240, line 13, change "a shaft and a tunnel drain" to a "shaft-and-tunnel drain."

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

THE PLANNING OF AERIAL PHOTOGRAPHIC PROJECTS

Discussion

BY F. J. SETTE

F. J. SETTE,[8] M. ASCE.[8a]—In asserting that the size of the annual photographic program is conducive to active competition and to lower prices, Mr. Norcross is quite correct, because it permits contractors to plan for continuous operation. It should be realized, however, that the size of the program by itself will not result in lower prices unless projects are scheduled in accordance with equipment available and the prevailing climatic conditions. The greater use of aerial photography for topographic mapping purposes and the need for rephotography because of cultural changes may well require a substantial, annual, aerial photographic program. This being so, planning, coordination, and integration, as now performed in the United States Department of Agriculture (USDA) for its own bureaus and agencies, should also be performed at a higher level, presumably in the Executive Office of the President (Bureau of the Budget) for the various departments of the federal government.

Large commercial concerns have been developing markets in foreign countries for some time, even before the war; and to the extent that they are successful and stay out of the domestic market, competition may be reduced. No American concern can safely stay out of the domestic market entirely.

In difficult terrain such as mountainous timbered areas, aerial photography may be considered more costly because of operational requirements. For example, the use of heavy planes is costly, narrowing the area of competition so that competitors may take advantage of this special circumstance. If an estimator knows the location of commercial concerns with reference to project areas, and the type of equipment at their disposal, he can determine a reasonable allowance for these circumstances.

Although the federal government is the largest user of aerial photography, the increasing use by others of the government's aerial photography, as reported

NOTE.—This paper by F. J. Sette was published in March, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: September, 1946, by T. W. Norcross, and James M. Cultice; and October, 1946 by Marshall S. Wright, and Robert H. Randall.

[8] Deputy Director, Bureau of Constr. and Field Operations, Civilian Production Administration, Office of Temporary Controls, Washington, D. C.

[8a] Received January 21, 1947.

by Mr. Wright, is gratifying. The writer feels that World War II interfered with the full utilization of the government's available aerial photography, by private interests. There should be an even greater demand in the future, especially if the photography is kept current with cultural changes. Mr. Wright states that one of the benefits outlined in the paper was that of enabling the contractor to utilize his personnel and equipment throughout the year by permitting him to move south in the fall. In the aerial photographic program of the summer of 1937, it apparently was the disappointing performance of a score of crews in the south that led the writer to an analysis of the monthly weather conditions. After 1937, large-scale immobilization of plane crews in the southern states in the summer and fall was no longer permitted.

Mr. Randall reports that the procedure of preparing estimates and of checking the bids against them is still followed, in general, not only by the USDA but also by other federal departments and establishments. If this procedure is continued in the future and if programs of various departments of the government are coordinated and integrated, extensive economies can be made by commercial concerns in planning their operations and in utilizing their equipment. Such economies, in turn, can be passed on to the government in the form of reduced bids.

The writer is particularly pleased that Mr. Cultice was able to give actual contract prices for aerial photography to compare with estimates developed in the two examples shown in the paper. Project A located in northern Missouri was estimated to cost from $1.90 to $2.00 per sq mile on the basis of 1938 prices. Mr. Cultice states that the contract price in 1940 for aerial photography in northern Missouri was $1.70 per sq mile. On September 19, 1946, bids were opened by the USDA, for aerial photography involving 13,484 sq miles in northwestern Missouri, southeastern Nebraska, and south-central Iowa. The low bid averaged $2.38 per sq mile. Project B was estimated to cost from $2.25 to $2.35 per sq mile. Mr. Cultice reports a contract price of $2.24 per sq mile in 1940. In 1946, the low bid for 8,430 sq miles (mostly in Indiana) was $2.42. One of the parcels in this area of 4,647 sq miles was bid at $2.35 per sq mile. On September 19, 1946, bids for a total coverage of 39,086 sq miles were requested, the low bid for which averaged $2.40 per sq mile. The project areas were located in Minnesota, Wisconsin, Iowa, Missouri, Nebraska, Illinois, Indiana, and Pennsylvania.

On December 30, 1946, bids were requested for a coverage of 17,178 sq miles in South Carolina, Georgia, and Florida; and the low bid averaged $2.53 per sq mile. For the September and December bid openings, the average of all the low bids was $2.44 per sq mile for a total coverage of 56,264 sq miles. Considering the scatter of the projects and the increases in cost of labor, materials, and supplies, the 1946 bids certainly compare favorably with the 1941 contract prices, which averaged $2.34 per sq mile for a coverage of 381,083 sq miles. It seems apparent that the gains made in the past are being maintained. Should wages and salaries keep increasing and the prices of supplies and materials keep pace, government technicians must give greater attention to each element of cost that enters into the contractor's bid in order to keep the costs of aerial photography as low as possible.

The writer is grateful to Messrs. Norcross, Cultice, Wright, and Randall for their contributions to his paper and for their favorable comments. The paper contained a number of assumptions, the validity of which could be upheld only by adequate cost records in the possession of commercial aerial photographers. Since the assumptions have not been challenged, the writer must conclude that these may not be too far "out of line" with actual costs. Contract awards for successive years, 1939 through 1941, and bids submitted in 1946, lend further support to the general reliability of the various assumptions.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

FACTORS CONTROLLING THE LOCATION OF VARIOUS TYPES OF INDUSTRY

Discussion

BY C. P. WOOD

C. P. WOOD,[7] ESQ.[7a]—Discussion, by amplifying the original paper, has emphasized some features that merit more attention. However, the discussion may be clarified by further definition of the underlying reasons for the location of industries: First, there are entirely new projects in which management is free to choose the best location; second, there are branches of existing industries which serve to increase capacity and to improve distribution facilities without affecting the operations of plants previously built; and, third, there are plants established for the purpose of decentralizing an industry.

Plants in the third class relieve the pressure on the parent plant and also provide the advantages mentioned in the case of branch factories. Distinction should be made between the migration and expansion of industries to take advantage of new opportunities and the decentralization of industries to avoid impairing or exhausting the resources of certain localities.

The New York economic report,[5] mentioned by Mr. Lewis, classifies industries dependent on a metropolitan location and discriminates between them and the industries needing more room than they can find in a great city.

It should be noted that New York (N. Y.) and other large cities are, in effect, industrial incubators for the less thickly populated sections of the United States. Many of the industries that started as small enterprises in the lofty buildings of great cities, which are the only places where they can find suitable rental space and skilled labor, later will expand into their own factories built either in the suburbs or at remote locations.

Reference to employment and industrial activity in Bridgeport, Conn., throughout the period which includes both World War I and World War II, is a good illustration of how planning in advance served to stabilize employment.

NOTE.—This paper by Charles P. Wood was published in March, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: November, 1947, by Harold M. Lewis, and R. F. Goudey.

[7] Industrial Engr., Lockwood Greene Engrs. Inc., New York, N. Y.

[7a] Received January 9. 1947.

[5] "Major Economic Factors in Metropolitan Growth and Arrangement," Regional Survey of New York and Its Environs, Vol. I, 1927, pp. 19-30 and 104-107.

Bridgeport is a highly developed manufacturing community with diversified industries and a large, well-housed colony of skilled labor. It is also a convenient distribution point for the great northeastern markets and especially for the New York metropolitan area. For these reasons, it is not comparable with southern or western cities of the same size or larger, which have comparatively few established industries and comparatively small colonies of skilled labor. Despite its many advantages, Bridgeport has found it profitable to have its resources studied with a view to revealing opportunities that otherwise might have been neglected. Absence of large sites was not allowed to obscure prospects for the development of smaller sites for industries particularly adaptable to conditions at Bridgeport.

Mr. Goudey's timely comment on the treatment of waste and the reclamation of water directs attention to the increasing importance of the water supply. It is not surprising that this comment should come from the West, where the conservation of water resources is of more consequence than it is in the East. However, the abundance of water east of the Mississippi River is no reason for neglecting the treatment of polluted waste or the conservation of pure water. Polluted streams have become characteristic of industrialized regions, because of lax enforcement or the absence of regulations in the early days of industrial development. Pollution of ground water resulting from spreading industrial waste on the land, also mentioned by Mr. Goudey, shows the necessity for treating dry waste as well as effluents. There are other familiar cases where polluted effluent is collected in pits, whence it gradually seeps into the ground, and where the resultant pollution may not be apparent until there is a need for pure well water in the vicinity.

Reclamation of sewage and other polluted waste water for boiler feed, cooling, and similar purposes will become more important as progress is made with the industrial development of arid sections and as the water supply is depleted elsewhere.

The suggestion that the location or relocation of industries could be facilitated by consulting local coordinating committees fails to take into account that the resulting publicity may nullify the committees' efforts. Competition among industries and localities interested in almost every industrial location prospect makes it necessary to keep the project confidential until a decision has been reached. In such cases, consulting engineers function because they can assemble information impartially without disclosing the identity of their client.

Similarly, the increasing importance of markets as a factor controlling location of industries requires the use of marketing data in appraising the characteristics of an industrial location. In this respect, however, engineers can get assistance from sales and distribution management.

A summary of new plant locations chosen since 1940[8] by six large companies manufacturing different products includes the following observations:

The total number of plants built or locations selected is eighty three, divided thus: Aluminum Company of America, 2; E. I. du Pont de Nemours and Company, 11; General Electric Company, 35; General Motors, 19; Philco, 7; and U. S.

[8] "Industry Fans Out," *Business Week*, November 23, 1946, p. 31.

Rubber Company, 9. Practically all the new locations are in states already heavily industrialized. Only twenty three are in cities larger than 100,000; the preference for smaller cities and towns being in the ratio of 3 to 1. The division by states, according to the number of plants added since 1940, is as follows:

New York	13	North Carolina	2
Ohio	12	Tennessee	2
Pennsylvania	8	Virginia	2
Indiana	7	Delaware	1
Massachusetts	7	Georgia	1
Illinois	5	Iowa	1
California	4	Kansas	1
Michigan	4	New Hampshire	1
Kentucky	3	Rhode Island	1
New Jersey	3	Washington	1
Texas	3	West Virginia	1

There is evidence of a tendency toward smaller plants. For example, the General Electric Company's thirty-five new plants are described as engaging from 30 to 1,500 employees each.

The controlling factors mentioned in the selection of these locations are labor, markets, transportation, raw materials, power, and fuel. Reasons given for changes from former locations include a continuing policy of decentralization, decentralization for better control of operations, social and economic benefits to employees, and lower costs.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852

DISCUSSIONS

EFFECT OF STRESS DISTRIBUTION ON YIELD POINTS

Discussion

BY R. K. BERNHARD

R. K. BERNHARD,[11] M. ASCE.[11a]—The "effect of stress distribution on yield points" described mainly from the point of view of the material itself is presented in this paper. It might be of interest to treat the subject more dynamically (60),[11b] discussing the change of equilibrium between specimen and testing machine or the effect of the natural frequency of test machines on the accuracy of indication in high-speed tests with respect to the determination of yield points. (The writer has contributed to this subject twice before (61)(62).)

In high-speed testing of essentially the same material, different results may be obtained depending upon whether a massive or a light testing machine is used. Furthermore, the stress-strain diagrams of certain materials indicate near the yield point peculiar forms, which in some cases may not be dependent upon the qualities of the tested material alone.

Static System.—Any testing machine with a hydraulic cylinder may be considered as a system composed of masses and springs. Masses constitute the frame of the machine (the crosshead, etc.) whereas springs with a definite elastic constant represent the test specimen and the hydraulic medium. At any moment there must be equilibrium between the summation of the elastic forces of the springs and the load on the ram of the machine.

In the case of a testing machine having a low elastic constant, a very small decrease of load will result with the increase of strain; a high elastic constant, on the other hand, will result in a considerable decrease in load with increase of strain. When plastic flow ceases, equilibrium between the two elastic forces (specimen and pressure medium) may be restored and the point of equilibrium again rises as soon as the pressure pump starts to force new liquid

NOTE.—This paper by F. G. Eric Peterson was published in April, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: September, 1946, by F. P. Shearwood, Edwin H. Gaylord, and L. J. Mensch; November, 1946, by George Winter, and O. Sidebottom and T. J. Dolan; and December, 1946, by J. F. Baker and J. W. Roderick.

[11] Prof. of Eng. Mechanics, Rutgers Univ., New Brunswick, N. J.

[11a] Received December 28, 1946.

[11b] Numerals in parentheses, thus: (60), refer to corresponding items in the Bibliography (see Appendix of the paper), and at the end of discussion in this issue.

into the cylinder. It must be borne in mind, however, that, with decreasing tension in the specimen, the plastic flow will also cease.

Any rapid flow gradually changes into some kind of creeping flow before equilibrium is finally achieved. Hence, in certain cases of high-speed testing such drop in the load-deformation diagram, due to plastic flow, may be independent of the quality of the tested materials.

Dynamic System.—The plastic flow or any pressure changes in the cylinder may take place very suddenly and the indicating device of the testing machine must follow immediately these rapid changes in both coordinates—that is, load and deformation. It is well known that as soon as rapid changes come into consideration the accuracy of any indicating device depends on the natural period of the entire vibrating system.

Any fluctuations during the test having a frequency of more than one tenth of the frequency of the machine will cause incorrect indications, assuming, for simplicity, that only vibrations of a sine form with a small damping factor come into consideration. This indicates, also, that different values for the upper and lower yield points, independent of the quality of the tested material and due to rapid changes of plastic flow near these yield points, may be found in certain cases of high-speed testing.

Effect Upon Yield-Point Determinations.—The surplus in tension at the upper yield point is a type of delayed phenomenon. Once plastic flow of the material in the specimen has started, the further shape of the diagram depends, at least to a certain amount, upon the character of the test machine. If the elastic constant of the test machine is low, any lower yield point will be suppressed, as the elongation of the test specimen produced by plastic flow means no decrease in tension for the testing machine itself. A very high elastic constant of the testing machine, however, will produce the well-known "leaping effect" fluctuations.

A similar result will be produced if the test specimen is long and has a low elastic constant, thus causing a heavy drop in the diagram. Consequently, a very short and rigid specimen may indicate no lower yield point at all.

Summary.—In the light of the foregoing comment the conclusions are reached that the upper and lower yield points are also associated with the characteristics of the testing machine and, to a certain degree, are independent of the quality of the material, and that in specific cases of high-speed testing the form of the specimen may be secondary.

Bibliography.—

(60) **"Einfluss der Federung der Zerreissmaschine auf das Spannungs-Dehnungs-Schaubild,"** by W. Späth, *Archiv für das Eisenhüttenwesen, No. 6*, December, 1935, pp. 277–283.

(61) *Journal of Applied Mechanics*, A.S.M.E., June, 1939, p. A-89.

(62) *Bulletin No. 106*, A.S.T.M., October, 1940, pp. 31–34.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

RIGID-FRAME STRUCTURES SUBJECT TO NONUNIFORM THERMAL ACTION

Discussion

BY CHARLES O. BOYNTON

CHARLES O. BOYNTON,[10] Assoc. M. ASCE.[10a]—The bending stresses in a closed continuous rectangular reinforced concrete frame, caused by heat applied to the inner face, have been analyzed in Part 2, to which this discussion appertains principally. Mr. Tommerup has ably demonstrated the excessive bending stresses inherent in such a frame, recommending that it be separated into sections by expansion joints.

It should be emphasized that the full magnitude of the uniform bending moment obtains in any part of the beam, however short, when restrained from bending. This condition may be illustrated by reference to Fig. 25, in which ABCD represents any very short part of the beam of length L and depth d, free from restraint. Side AB has been heated $t°$ F hotter than side CD, with relative elongation ΔL. If the temperature coefficient of linear expansion is α, then $\Delta L = \alpha\, t\, L$ and the central angle formed by the tangents to the ends of the elastic curve, $\theta = \frac{\Delta L}{d} = \frac{\alpha\, t\, L}{d}$ (θ being very small).

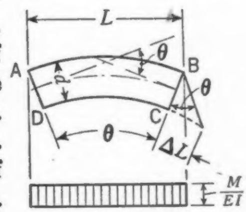


FIG. 25

Let the conjugate beam be loaded with the $\frac{M}{E\,I}$-diagram, M being constant since the curvature is uniform and just sufficient to "straighten" the curved beam. The area of this diagram, $\frac{M\,L}{E\,I}$ by the moment area method equals the angle θ, or $\frac{M\,L}{E\,I} = \frac{\alpha\, t\, L}{d}$ and $M = \frac{\alpha\, t\, E\, I}{d}$. Substituting the author's values, $M = \frac{0.0000079 \times 400 \times 2{,}000{,}000 \times 14{,}230}{20} = 4{,}500{,}000$ in.-lb. This bend-

NOTE.—This paper by Carl C. H. Tommerup was published in June, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: November, 1946, by I. Oesterblom, and Frank R. Higley; and January, 1947, by A. L. Miller.

[10] Engr., Structural Design, Blanchard & Maher–Frederic R. Harris, Inc.–Keller & Gannon, San Francisco, Calif.

[10a] Received January 2, 1947.

ing moment agrees with the author's determination and is about twice the safe resisting moment of the beam. However, had the more common grade of concrete been employed, with E equal to about 3,000,000 lb per sq in., the resulting stresses would be 50% greater.

In view of the extremely large uniform bending stress in this design, it would be logical to destroy the bond of the tensile steel by applying a coating, thus allowing the concrete to crack at very frequent intervals. The tensile bars would help to prevent disintegration of the concrete.

If the ends of the frame are restrained, there will be, in addition to the bending stress, an axial thrust due to an average temperature rise of 200° F, the unit compressive stress being $E\,\alpha\,t = 2{,}000{,}000 \times 0.0000079 \times 200 = 3{,}160$ lb per sq in.

The stresses developed in this example are so far beyond safe values that some basic changes in the design would be indicated, such as protecting the hot face of the concrete with firebrick and ventilation, in addition to providing expansion joints.

This paper affords good information on a type of design which is frequently not given careful consideration.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

TRUCK SPEED AND TIME LOSS ON GRADES

BY ELLIOTT J. DENT, AND F. B. OGLE

ELLIOTT J. DENT,[10] M. ASCE.[10a]—In discussing this interesting subject there are certain practical considerations that the writer believes should receive more emphasis.

On the upgrades encountered on main highways the modern passenger car can, ordinarily, maintain normal speed; in fact, there is often a considerable reserve of power for pickup. Trucks, on the other hand, are slowed down to a marked degree. On the downgrades trucks can, and do, maintain the legal or safe speed limits.

On an upgrade a car can readily slow down or stop; on a downgrade the control of the vehicle is much less efficient. Other things being equal, the safe speed on an upgrade is greater than that on a downgrade.

If the foregoing were the only considerations, the easy and safe places for passenger cars to pass trucks would be on the upgrades. It is unfortunate that in such locations the summits limit the sight distances and make it unsafe to pass, whereas on the downgrades the visibility is much better.

In rolling country, on heavily traveled two-lane highways, it is inconvenient and most exasperating that, when the safe and available speeds are favorable for passing, the visibility so often inhibits such action. On the other hand, when the visibility is satisfactory, the available relative speeds are unfavorable.

In the majority of cases the logical solution is the provision of an additional traffic lane on the upgrade side of the crest.

F. B. OGLE,[11] Assoc. M. ASCE.[11a]—The proper maximum grades for highways, required to meet a specific set of conditions, can be determined by the long-needed method presented in this paper. Certain variable factors, particularly driver habits and truck operating characteristics, will be minimized by driver training and by the standardization of truck building and maintenance, which will be forced by competition. This development is foretold by the

NOTE.—This paper by J. W. Stevens was published in September, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: December, 1946, by Jonathan Jones.

[10] Cons. Engr., Washington, D. C.

[10a] Received December 30, 1946.

[11] Senior Resident Engr., Texas Highway Dept., Lubbock, Tex.

[11a] Received January 20, 1947.

experience in railroad operation of the past. The method proposed by Mr. Stevens is sound, simple, and easily applied.

The first designers of the modern highway were primarily concerned with providing an all-weather surface for light, self-propelled vehicles of relatively low speeds. The alinement usually followed existing roads; and the grade line followed the existing grade closely. Based on present-day standards, this alinement was poor, and the adverse grades were excessive in many cases.

As the traffic load increased, improvements consisted primarily of widening pavements, inserting curves at angle turns, flattening existing curves, and reducing excessive grades to an assumed maximum rate of rise. The assumed maximum grade rates were low enough to avoid any serious reduction in speed of the then predominately automobile traffic. Highway traffic capacity did not appear as a problem.

The appearance of the freight hauling truck in substantial numbers in the past few years has changed the picture materially; and highway capacity has become an important problem in highway design. The trend clearly indicates that in the near future, certainly within the normal life span of the highways now being designed, truck traffic will predominate. This trend will be accelerated by the adoption of uniform traffic regulations by all the states. Such regulations will come in time, and their adoption will increase interstate movement greatly. For economic reasons, commercial trucks in interstate haul must operate with heavier loads and at higher speeds.

Grades that can be negotiated by the passenger automobile at 50 miles per hr or more will force the trucks into lower gears, requiring them to crawl up at 10 miles per hr, or less. On long grades where traffic is heavy, this produces a serious condition, both from the standpoint of safety and loss of highway capacity. This is apparent to all drivers. One method proposed by Mr. Stevens for relieving this congestion is to provide passing lanes which will release the faster moving vehicle; but it will give no relief to the trucks, which can be relieved only by proper grade design with the truck as the controlling factor.

Grade design based on truck operation will increase the construction or first cost; but it will reduce the unit cost of truck operation and increase the highway capacity. The net result will be to reduce the actual cost based on number of vehicles accommodated; and these are, in fact, the true costs.

It appears that the expenditure of public funds for the convenience of trucks operated for private profit is not justified, as has been contended by the railroad companies. This is unquestionably true in so far as the benefits accrue to the truck operator; but the movement of freight on the highways has become vital to the economic life of the United States, and any reduction in the cost of freight movement will accrue to the public.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

EXPERIMENTAL OBSERVATIONS ON GROUTING SANDS AND GRAVELS

Discussion

BY WILLIAM S. FOSTER

WILLIAM S. FOSTER,[17] Assoc. M. ASCE.[17a]—While a sanitary engineer may have been searching diligently for cross connections with an unapproved water supply in some remote corner of his distribution system, a more serious "cross connection" may have escaped notice under his very nose. This is the challenging observation presented in the Machis' paper. In well drilling the customary practice is to seal off the undesirable aquifers by producing a barrier at the casing. Nevertheless, as Mr. Machis states, and as the experience at Baltimore (Md.) shows, this undesirable water can find its way down the outside of the casing and thereby pollute the new aquifer. At Baltimore the pollution consisted of salt water which renders the supply unfit for further use.

Mr. Machis' system, borrowed from the petroleum industry, consists of pumping a cement slurry around the outside of the well casing and forcing it under pressure against the wall of the well. In coarse sands and gravels the cement will penetrate until the internal friction of the sand grains halts the intrusion of the slurry and allows it to block and seal off the aquifer. In fine sands the slurry will merely collect like a filter cake, and the pressure applied will squeeze out the excess water leaving a firm, strong cement barrier with a consistency of from 4 gal to 5 gal per sack. In its underground location, this barrier will have an even temperature; it will be constantly in contact with moisture; and, probably, will not be subjected to variations in pressure. Consequently, for practical purposes, it should enjoy virtually an indefinite life. As Mr. Machis states, by varying the consistency of the slurry, one should be able to seal long lengths of the well casing or to apply local applications of grout and seal off a limited area.

The fear that this application of a thin grout (often preceded by water flushed up and outside the well casing in order to establish circulation) may disturb the sand and gravel strata and make them slough against the side of the

NOTE.—This paper by Alfred Machis was published in November, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1947, by James B. Hays.

[17] Eng. Editor, *The American City Magazine*, New York, N. Y.

[17a] Received January 23, 1947.

well casing, thus blocking the operation, seems unfounded. This is particularly true for wells drilled by the rotary process in which the wall of the wells is held firm by the driller's mud. In most cases the pressure of the overburden and the precipitated minerals will consolidate the sands and gravels sufficiently to make them hold their positions firmly. Actual drilling logs show that clays and shales with less internal strength slough away more readily than do sands and gravels.

One question that arises in the mind of the uninitiated (and most engineers who are not petroleum specialists fall in that class) is whether some materials other than cement might give better results. Of course, a beam or briquette made of neat cement will give an excellent account of itself on a strength basis; but it does insist on shrinking, even under favorable conditions. The writer recalls at one time helping to waterproof a curing room in a laboratory in which both temperature and humidity were to be maintained within close limits. A neat-cement layer of grout applied to the brickwork in the room shrank and cracked sufficiently to lose its effectiveness. It may be that shrinkage would not be a factor in the case of well grouting; but water has an admirable trait of persistency and an ability to enter any small opening and develop it to amazing proportions. It would seem that some other product like asphalt or a dense impervious clay, which does not shrink when temperature and humidity are constant, would do the same sealing operation equally well and would eliminate this cracking danger.

Another point that immediately comes to the attention of the practicing engineer is how he can apply this useful grouting operation in his own installations. Evidently this sealing work is not a task for amateurs. Since it is being performed in the petroleum field, there must be concerns that would be willing to do the same work in the water works field. Consequently, the water works engineer would like to know what specifications he should call for in order to encourage these concerns to bid, and at the same time protect the interests of his client.

On the basis of studies by Mr. Machis, the painstaking engineer may very well insist that when sealing operations are required in well construction, they be accompanied by grouting the well casing in that particular area.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
DISCUSSIONS

RECHARGE AND DEPLETION OF GROUND-WATER SUPPLIES

Discussion

BY E. A. VAUBEL

E. A. VAUBEL,[3] Assoc. M. ASCE.[3a]—The thought that depletion of ground-water supplies does not occur suddenly but results from hydrological withdrawal from storage at rates greater than recharge is strikingly presented in this paper. Because many variables enter into a decline in water levels and differ somewhat even at two points in a restricted area of a few square miles, it is reasonable that many people might assume that declining levels would spell exhaustion of ground-water supplies within a few years.

As the author states, the properties of the individual aquifers determine how much water can be drawn from them by wells. When progressive declining levels occur due to a concentrated pumping requirement in a limited area, it is certainly proper to consider and to put into effect certain engineering concepts which promote effective utilization and conservation of ground-water resources. These concepts are:

1. Proper well spacing, projected by use of the Theis nonequilibrium formula applied to measured production and pumping levels of existing adjacent wells and pilot wells, can enable large municipal and large industrial users to secure their ground-water requirements without fear of depletion.
2. Sometimes in an existing well field, a redistribution of pumpage to equalize the effect of the water levels will help to forestall a rapid decline in a given sector of the field. This is attained by taking production from outlying wells in order to enlarge and equalize the withdrawal per given unit of area. The piezometric surface then tends to be devoid of the depression cones created by excessive pumpage in a small area in a well field. In terms of operation procedure, it may be stated that the quantities withdrawn from outlying wells should be equivalent to those withdrawn from strategically located wells in the center of the well field.

NOTE.—This paper by Charles L. McGuinness was published in September, 1946, *Proceedings*.
[3] Engr., The Layne Texas Co., Ltd., Houston, Tex.
[3a] Received January 31, 1947.

3. In many areas additional exploration will reveal the exact depth and thickness of aquifers and permit withdrawal of water for piping from a relatively virgin area to the point of use, without undue economic expense. A planned test drilling program is advisable when considering a long range ground-water supply requirement so as to be certain that the extent of the aquifer is definitely known.

4. Where known sands afford water of different mineral quality, it is expedient and within good conservation practice to utilize the better quality of water for boiler and domestic use. By this is meant water of low hardness and with chlorides generally within the United States Public Health Service standards. Where supplies of greater hardness and chloride content occur, these can generally be better utilized for certain industrial processes and cooling.

5. In serving the water supply needs of a large community the fact must not be overlooked that certain surface streams may supplement a ground-water supply program or vice versa. Sometimes there is a tendency to balance production, storage, and distribution costs of one against the other, although both may be required to supply the present and future needs of a large and growing center of population adequately. Surface and ground water each have inherent advantages and disadvantages; the advantages of each are sometimes necessary to complete the water supply picture.

The points outlined, which are suggested procedure for attacking a ground-water supply depletion problem, were covered in part by the author, but item 5 supplements his presentation. Delineation by items serves to emphasize the points covered in detail and by example in the paper.

The Houston, Tex., area presents an example of an aquifer which has a high rate of recharge due to heavy precipitation, a large outcrop area with average permeability, and a series of sands totaling approximately 600 ft in thickness; and which, as a result, affords a large perennial supply. However, the discharge required from the aquifer is becoming increasingly greater due to industrial and municipal growth and irrigation usage. Therefore, the previously enumerated factors will have to be utilized to provide ample quantities for future water supply requirements.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

CLEANING AND GROUTING OF LIMESTONE FOUNDATIONS, TENNESSEE VALLEY AUTHORITY

A SYMPOSIUM

Discussion

BY BERLEN C. MONEYMAKER, AND C. E. BLEE

BERLEN C. MONEYMAKER,[7] Assoc. M. ASCE.[7a]—The two papers in this Symposium illustrate the magnitude of problems encountered in the construction of dams on limestone foundations in a region that has been humid for a long time, geologically. The rocks of the Tennessee Valley have been exposed to the relentless attacks of the agents of weathering for millions of years. Siliceous rocks have decayed to great depth and the carbonate rocks (limestone, marble, dolomite, and calcareous shale) have weathered by solution. Soluble rocks not only have been "worn down" to their present elevation largely by solution, but they are also "honeycombed" by caves and smaller solution channels to considerable depths. Cavities are largest and most numerous in the upper part of the bedrock, becoming smaller and less numerous with increasing depth.

Of the sixteen dams completed by the Tennessee Valley Authority (TVA) to date, nine are founded entirely or partly on limestones. Three additional dams in the area, built before the creation of the Authority, are also founded on limestone foundations. Although some of these sites were much better than others, all of them were to some degree cavernous.

Mr. Pauls' paper on Fort Loudoun Dam is a fairly complete record of difficulties met during the construction period. The first geologic problem involving this project was the selection of a dam site. A preliminary geologic investigation of that section of the Tennessee River in which Fort Loudoun was to be located, undertaken in May, 1935, soon revealed that no really good dam sites were to be found. To locate the best site available, a detailed study was made of the river from mile 576 to mile 609. The geology of this 33-mile

NOTE.—This Symposium was published in December, 1946, *Proceedings*.

[7] Chf. Geologist, TVA, Knoxville, Tenn.

[7a] Received January 30, 1947.

stretch of the river was carefully mapped and fourteen potential dam sites were selected for investigation.

Some of these sites (see Table 4(*a*)) had been considered by the U. S. Army Engineers; others had not. A few holes drilled by the engineers at the Marble Bluff site showed the rock to be cavernous. The numerous sinks at both Marble Bluff and Bogart Island indicated extensive bedrock solution. At the Loudon sites, foundation conditions were thought to be fairly good but the reservoir rim was low, thin, and cavernous. The Rock Quarry Bar sites were considered the best dam sites in the entire 33-mile stretch of the river; but they were also ruled out because of the low, thin, and cavernous reservoir rim. The Bussell Shoals sites and the Coulter Shoals site were soon found to be on deeply cavernous rock.

TABLE 4.—SITES INVESTIGATED

Site	River mile
(*a*) FORT LOUDOUN DAM	
Marble Bluff [a]	577.43
Bogart Island [a]	583.75
Loudon:	
Site No. 1 [b]	592.06
Site No. 2 [c]	592.23
Rock Quarry:	
Bar No. 2 [c]	594.72
Bar No. 1 [d]	594.83
Lenoir Shoals [e]	601.05
Lenoir Ferry [e]	601.65
Belle Canton Island [f]	602.30
Bussell Shoals: [a]	
Site No. 3	603.30
Site No. 2	603.40
Site No. 1	603.54
Sister Island [a]	605.03
Coulter Shoals [a]	608.30
(*b*) DOUGLAS DAM	
Cain Island [g]	19.73
Kykers Ferry [h]	28.87
Douglas [h]	32.28
Dandridge [h]	44.81

[a] Knox dolomite limestone. [b] Tellico sandstone and shale. [c] Tellico sandstone and limestone. [d] Tellico shale. [e] Chickamauga, Holston, and Tellico limestones. [f] Tellico limestone, Sevier shale and limestone. [g] Rutledge limestone. [h] Knox dolomite.

When the operating level of Watts Bar Dam was established at El. 745, several of these sites were automatically dropped from consideration and the search for the Fort Loudoun site was thus limited to a 12-mile stretch of the river. Exploratory drilling at four sites which had not been ruled out because of geologic or engineering reasons revealed that the Belle Canton Island was the best site available. This site was therefore selected as the site for Fort Loudoun Dam.

The rock involved in the foundation of Fort Loudoun is nonuniform in character. It ranges from the coarsely crystalline sandy limestone of the Tellico formation, through coarsely crystalline pure limestone, finely crystalline impure and shaly limestone, to calcareous shale in the Sevier formation. Structurally, the strata of both formations have an average strike of about N 42° E and dip southeastward at angles of from 30° to 35°. The limestones of both formations are cut by numerous steep to vertical joints and the shaly members of the Sevier are somewhat folded (see Fig. 3) but less extensively jointed. In both formations, the limestones have been dissolved away along all structures sufficiently open to permit the movement of ground water. The shales of the Sevier formation are not soluble; but in many places the soluble calcium carbonate has been leached from them, leaving seams of soft mud. Although such mud seams are not cavities, they are actually little different from mud-filled solution cavities as foundation defects.

From the beginning it was recognized that the Fort Loudoun site contained serious defects of considerable magnitude. Even while the dam site was under

exploration, the chief geologist, the late Edwin C. Eckel, Affiliate ASCE, appraised it in the following words: "The Belle Canton Island [Fort Loudoun] dam site is much worse than Chickamauga." The correctness of this appraisal was verified by the Authority's Board of Consultants, whose report of June 18, 1942, stated that "No site of the whole Tennessee Valley System has shown so many weaknesses."

Mr. Pauls' paper is a detailed account of the nature of these numerous defects and the methods by which they were overcome to afford excellent foundation conditions for Fort Loudoun Dam.

Mr. Taylor's paper on Douglas Dam deals with a project in which the problems presented by a limestone foundation were entirely different from those at Fort Loudoun. The only point of similarity between the two projects was that the problems at both were the result of solution of the bedrock.

The selection of a site for Douglas Dam was a comparatively simple matter. Before the final choice was made, a 30-mile stretch of the river was considered. Many localities were easily ruled out on basis of topography and the general poorness of foundation conditions, but four sites were considered in some detail. These sites are listed in Table 4(*b*).

Preliminary geologic field studies revealed that the rock at all these dam sites was cavernous, but the Douglas site appeared to be much better than the other three. It met the engineering requirements somewhat better than the other sites and was selected as the site for Douglas Dam.

Douglas Dam is located on a rather narrow belt of Knox dolomite, which is exposed along the northwest limb of a broad, relatively flat syncline. At the dam site, the strata strike from N 40°–60° E and dip southeastward at angles of from 15° to 20°. The Knox dolomite consists mainly of cherty dolomite and limestone, although there are some sandy beds in the formation. Except for the sandy layers and the chert, the formation is of fairly uniform solubility throughout.

The preliminary drilling proved that the upper 20 ft to 60 ft of bedrock was extremely cavernous, especially in the original channel and flood plain areas, and that a sizable open cavity existed in each abutment. Subsequent excavation revealed that the cavities were developed along joints and bedding planes. The two large abutment caves and many of the smaller cavities were developed along both joints and bedding planes, but some of the smaller openings were restricted to one structure or the other.

The bedding-joint cave in the right abutment was a "blowing cave." It was intercepted by a drill hole in the early stages of exploration and the "blowing" phenomenon was noticed by the drillers. Two other holes in the same general area exhibited like behavior. In the summer, these holes became "sucking" holes. The "blowing and sucking" behavior of the drill holes suggested that the cave was extensive and proved that somewhere it had a natural opening, although such an opening was never found. The principal problem presented by the two abutment caves was that of watertightness; however, there was no definite assurance that the 40 ft of sound rock overlying the right abutment cave afforded adequate support for the end of the dam. Although these caves required a considerable amount of painstaking work under none too

pleasant working conditions, their treatment did not involve any difficult or unusual problems. They were cleaned out and then plugged with concrete. The cave in the south abutment, in some places, was a rather crooked channel, and it was found to be more economical to mine out enough rock to provide for a straight cutoff plug instead of following the meander-like bends of the natural opening.

As indicated in Mr. Taylor's paper, the greatest problem presented by the cavernous rock at Douglas was that of controlling the water in the cofferdams. The strike of the strata was essentially parallel to the river channel. Solution cavities developed along bedding planes and strike joints served as conduits to pipe water into the cofferdams. All attempts to shut the water out of the cofferdams were ineffective, and excavation progressed with difficulty, block by block. Pumps in constant operation discharged 100,000 gal per min for some time, the maximum discharge being about 105,000 gal per min.

In spite of the tremendous problem of controlling water, an excellent watertight foundation was prepared. The transition between highly cavernous rock and tight, noncavernous rock was nearly everywhere sharp. The depth to which solution had progressed varied from place to place but there was no relation between the depth of solution and the lithologic character of the rock.

The two papers form a valuable record of the geologic problems, and their solution at dam sites in limestones. Mr. Taylor's paper on Douglas Dam is especially interesting as a record of a hard-fought and desperate struggle between engineer and river under the handicap of a short construction schedule. At times the outlook was dark for the engineer, who finally won by dint of his sound experience with limestone foundations in general and his very thorough knowledge of the geologic conditions at Douglas Dam in particular.

C. E. BLEE,[8] M. ASCE.[8a]—The paper dealing with the Fort Loudoun Dam is of particular interest to the writer since he was directly in charge of the construction of this dam as project manager for the first eighteen months, being succeeded by J. K. Black, M. ASCE, who had been construction engineer on the project. It is a very comprehensive and well-presented account of the complex foundation conditions encountered and of the procedures employed to overcome these difficulties.

Perhaps the most significant feature of the foundation conditions at the Fort Loudoun site was the comparatively great depth to which the solution channels extended. This is stressed in the paper as is also the fact that structural deformation was often the determining factor which controlled the depth and extent of disintegration rather than the solubility of the rock. This latter effect was readily discernible on the ground from the crisscrossed lines of disintegration following bedding planes, joints, and faults and always having an enlarged zone of disintegration at the intersection of these features. In the case of a flowing stream where differences in head are available, there is, of course, no reason, from the standpoint of hydraulics, why water should not find its way through open cracks or fissures thus producing solution at considerable

[8] Chf. Engr., TVA, Knoxville, Tenn.

[8a] Received February 17, 1947.

depths. At some dam sites there are geological conditions which definitely limit the depth to which solution can go. For instance, at the site of the Chickamauga Dam, there is a stratum of bentonite which effectively stopped the penetration of water and so afforded a horizon below which it was known that foundation treatment need not be carried.

At Fort Loudoun one of the chief difficulties from the construction standpoint was in cutting off the flow of water into the excavation areas. Nearly 700,000 cu ft of grout, including neat cement, clay cement, and asphalt, were required to control the flow through the rock into the first-stage cofferdam. In sinking the calyx holes within the cofferdam area, it was often necessary to encircle the hole with a line of supplementary grout holes to cut off the flow of water through seams encountered at greater depth.

Only by the utilization of the experience gained and of devices developed during comparatively recent years could a major dam and lock structure be successfully constructed on a foundation as bad as that which existed at the Fort Loudoun site. At Fort Loudoun the engineers of the Tennessee Valley Authority (TVA) had the benefit of previous experience and of the devices developed at the comparatively large number of dams, including Norris, Chickamauga, Guntersville, Kentucky, and Hales Bar, which had been built on difficult limestone foundations—as well as the experiences of other organizations recorded in the technical literature. By the publication of this Symposium together with those previously published in the *Transactions* of the Society, the experience gained by TVA is made available to the profession as a whole.

Effectiveness of the cutoff and grouting operations at Fort Loudoun is demonstrated by the low rate of seepage past the dam site. Holes 3 in. in diameter were drilled just downstream from the cutoff on approximately 11-ft centers throughout the spillway section. These open drainage holes extend from 30 ft to 80 ft into the rock beyond the excavated surface and so can be expected largely to intercept any seepage through the foundation rock under the concrete structure. The seepage, as measured by the discharge from the total of these holes, amounts to less than 10 gal per min with a full reservoir. Careful records have been kept of the ground-water levels in both abutments of the dam and these indicate that there is very little seepage through the abutments.

As stated by Mr. Pauls in the "Summary," the operations for consolidation and for an effective cutoff challenged the combined efforts of the designing engineers, field engineers, geologists, and construction personnel. The close cooperation that existed between the engineers, geologists, and construction personnel was an outstanding feature of the Fort Loudoun job; that the challenge presented by the difficult foundation conditions was successfully met is attested by the results achieved.

A complete and explicit account of the methods and operations used in foundation treatment at Douglas Dam is given by Mr. Taylor. The information thus made available should be useful to anyone having a similar foundation problem.

Cavities encountered in the foundation rock at Douglas Dam were, in general, larger and more continuous than those at Fort Loudoun but were not as complex nor did they extend to such great depths. At Douglas, blowouts and

leakage through cavities into the cofferdam areas presented particularly severe problems because of the extremely fast construction schedule for the job—calling for completion in less than fourteen months after work was started.

Use of large-diameter calyx drill holes as construction devices to give working access to disintegrated seams or cavities underlying sound rock, and to afford a means of depositing concrete in these seams after being cleaned out, is well illustrated in the work described. Another and rather unusual device was the use of a grout curtain to confine, to the desired area, the grout used for consolidation.

As in the case of Fort Loudoun, the measurements of foundation drainage and the ground-water investigations at Douglas Dam demonstrate the effectiveness of the foundation treatment.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

STRENGTH OF BEAMS AS DETERMINED BY LATERAL BUCKLING

Discussion

BY NEIL VAN EENAM

NEIL VAN EENAM,[30] M. ASCE.[30a]—Two separate phases of the problem of lateral buckling of beams and girders are treated in this paper. In the first part, additional values of the factor k are derived for various types and conditions of loading which supplement the factors m derived by Professor Timoshenko.[2] The derivation of these additional factors k, which can readily be converted to the Timoshenko factors m, is a valuable contribution. This part of the paper will be appreciated by those who prefer original sources for data on the solution of this type of problem. It is unfortunate, however, that for braced beams of two bays, the symbol l denotes the full length of the beam between end supports in Fig. 8 and in Eqs. 42 and 49, whereas in Table 1 the symbol l denotes the unsupported length of the compression flange, which in this case is one half of the full length of the beam. This may lead to some confusion.

Simplified formulas, both for critical stresses and for working stresses, are proposed in the second part of the paper. These formulas employ the dimension ratio $\frac{l\,d}{b\,t}$. In Figs. 10 and 11, results obtained by these simplified formulas are compared with those obtained by more exact methods. An analysis of the principal source of variation in results follows the comparison.

Curves ABCD of Figs. 10 and 11, representing the more exact values for the type of loading shown, were computed according to Professor Timoshenko's method.[31] However, identical results would have been obtained by using Eq. 4 of the paper. The torsion constants K entering into the more exact

NOTE.—This paper by Karl de Vries was published in September, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: December, 1946, by George Winter, and David B. Hall; and February, 1947, by Theodore R. Higgins.

[30] Highway Bridge Engr., U. S. Public Roads Administration, Washington, D. C.

[30a] Received January 27, 1947.

[2] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York and London, 1936, Chapter V, p. 239.

[31] *Ibid.*, p. 268, Eq. 164, and p. 278.

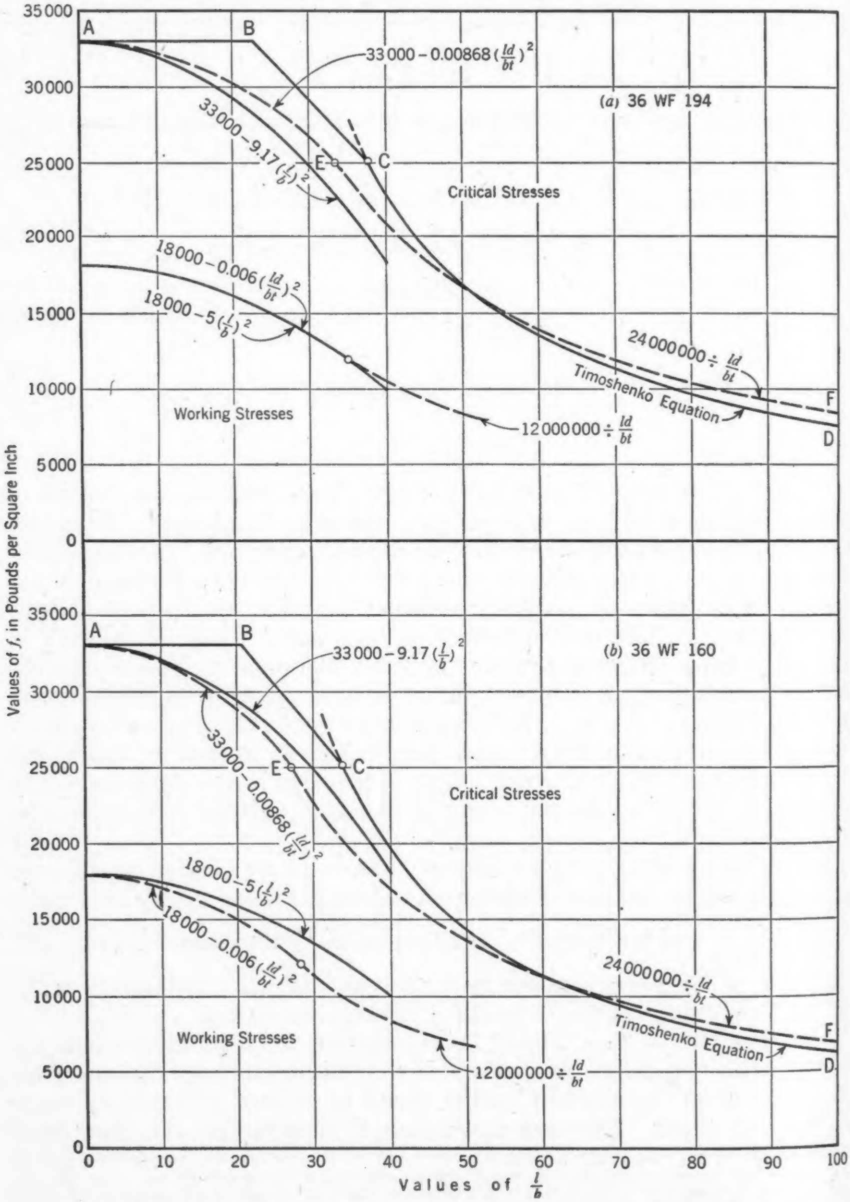


FIG. 10.—COMPARISON OF FORMULAS FOR CRITICAL AND WORKING STRESSES IN TYPICAL ROLLED BEAMS, UNIFORM LOAD APPLIED AT THE CENTROID

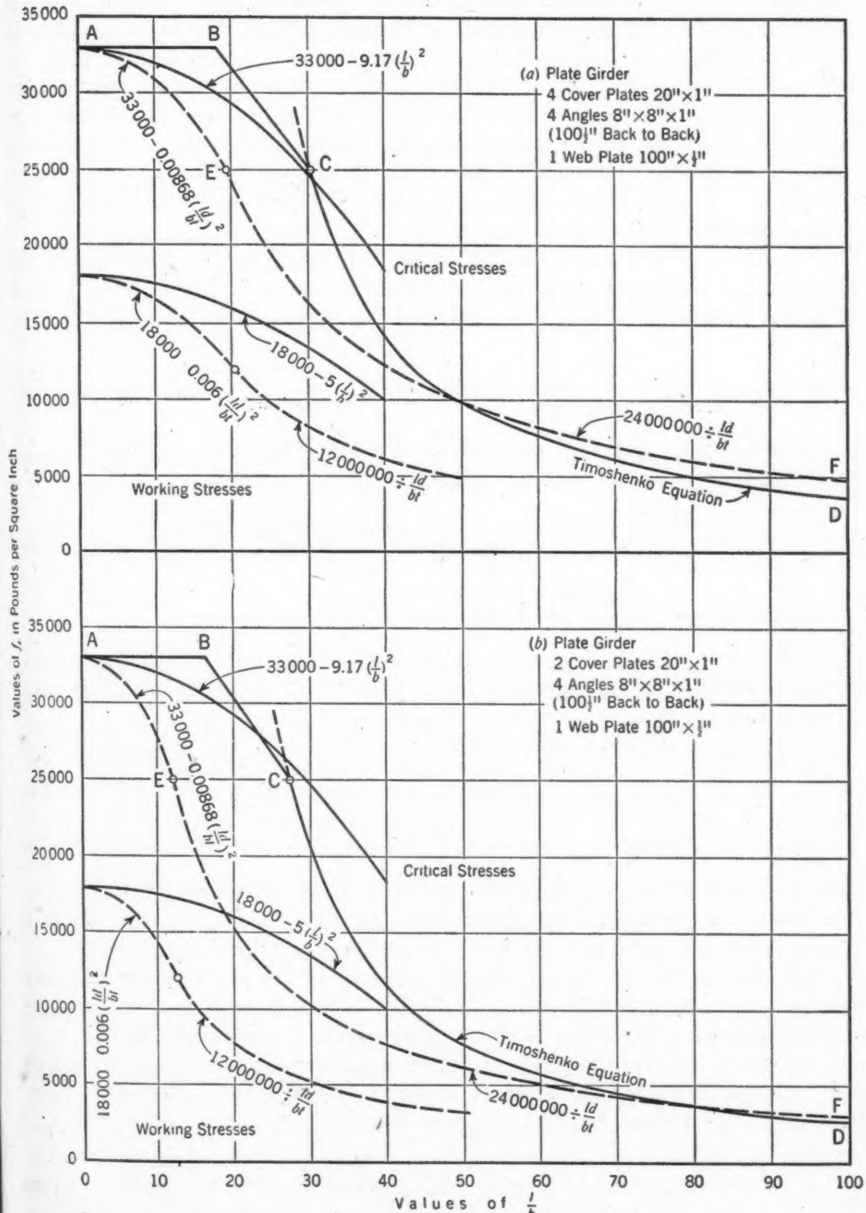


FIG. 11.—COMPARISON OF FORMULAS FOR CRITICAL AND WORKING STRESSES IN TYPICAL PLATE GIRDERS, UNIFORM LOAD APPLIED AT THE CENTROID

formulas may be accurately determined for rolled beams.[9] In riveted plate girders, there may be considerable slip between the various parts comprising the section, and there is uncertainty as to the proper reduction to be made on this account. The tests made by I. E. Madsen,[11] Jun. ASCE, appear to give the best data available, and it was decided to follow his recommendation that the values of K for riveted plate girders be taken equal to one third those for identical monolithic sections. Curves ABCD of Figs. 11(a) and 11(b) may therefore be slightly on the conservative side. There is need of further tests to confirm Mr. Madsen's findings.

Curves AEF of Figs. 10 and 11 were computed by the formulas of the paper, using constants consistent with those given in Table 4(a) for carbon steel and loads applied on the centroids of the sections. Values derived from:

$$f = 33{,}000 - 9.17 \left(\frac{l}{b}\right)^2 \quad \text{(66)}$$

are also included. Eq. 66 was derived by multiplying the commonly used formula for carbon steel working stresses,

$$f = 18{,}000 - 5 \left(\frac{l}{b}\right)^2 \quad \text{(67)}$$

by the safety factor $\dfrac{33{,}000}{18{,}000} = 1.83$.

The plate girder of Fig. 11(a) is of the same section as that used for illustrative purposes in Part III of the paper. A similar section, but with two cover plates instead of four, is included in Fig. 11(b).

In all cases, both of rolled beams and of plate girders, at the higher ratios of $\dfrac{l}{b}$, Mr. de Vries' formulas yield results very close to those given by Professor Timoshenko. At the lower ratios of $\dfrac{l}{b}$, however, values proposed by Mr. de Vries are considerably less than those by Professor Timoshenko. Furthermore, the transition curves AE appear to drop off too rapidly, for the points E should practically coincide with the points C on the Timoshenko curves.

The reasons for this characteristic property of the de Vries formulas can best be shown by a mathematical analysis. The notation used in the paper will be adhered to, but the list of symbols given in Appendix II should be supplemented by the following: m equals a factor whose value is dependent upon the type of loading; μ equals Poisson's ratio, whose value for steel is 0.30; and a is a torsion bending constant equal to:

$$a = \frac{d}{2}\sqrt{\frac{2(1+\mu)\, I_y}{K}} \quad \text{(68)}$$

[9] "Structural Beams in Torsion," by Inge Lyse and Bruce G. Johnston, *Transactions*, ASCE, Vol. 101, 1936, p. 857.

[11] "Report of Crane Girder Tests," by I. Madsen, *Iron and Steel Engineer*, November, 1941, Section 4, p. 68.

The general equation for the critical value of the bending moment at which buckling occurs is:[2]

$$M = \frac{m \sqrt{B\,C}}{l} \qquad \text{(69)}$$

It is convenient to consider a beam subjected to pure bending, since for pure bending the value of the factor m is simply:[32] $m = \pi \sqrt{1 + \pi^2 \frac{a^2}{l^2}}$.

Making this substitution for m, the value of the critical compressive fiber stress is:

$$f = \frac{M\,y}{I} = \frac{d}{2\,I_x}\,\pi \sqrt{1 + \pi^2 \frac{a^2}{l^2}}\,\frac{\sqrt{B\,C}}{l} \qquad \text{(70)}$$

Eq. 70 may be expanded and written:

$$f = \frac{E}{2}\left(\frac{\pi\,d}{l}\right)^2 \sqrt{\left(\frac{l}{\pi\,d\,I_x}\right)^2 \frac{I_y}{2}\,\frac{K}{(1+\mu)} + \left(\frac{I_y}{2\,I_x}\right)^2} \qquad \text{(71)}$$

If the second term under the radical is dropped, Eq. 71 becomes:

$$f = \frac{E\,\pi\,d}{2\,l\,I_x} \sqrt{\frac{I_y\,K}{2\,(1+\mu)}} \qquad \text{(72)}$$

Consider now a symmetrical rolled beam section, consisting of a web plate and two flanges. In such a section, the effect of the web on the moments of inertia I_x and I_y and on the torsion constant K is small and may be neglected. Approximately, then,

$$I_x = \frac{b\,t\,d^2}{2}; \quad I_y = \frac{b^3\,t}{6}; \quad \text{and} \quad K = \frac{2\,b\,t^3}{3} \qquad \text{(73)}$$

Substituting these values in Eq. 72 and also evaluating E, μ, and π,

$$f = \frac{19{,}483{,}000}{\dfrac{l\,d}{b\,t}} \qquad \text{(74)}$$

Eq. 74 is of the type proposed in the paper. It should be recalled that, in its derivation, the second term under the radical in Eq. 71 was dropped. By comparing curves (*b*) and (*c*) of Fig. 12, the effect of dropping this term may readily be seen. The curves were computed for a 36WF160 beam. Curve (*a*) represents the critical stresses in a beam loaded uniformly along the centroid of the section over its entire length. In curve (*b*), critical stresses are shown for a beam subjected to pure bending. The ordinates of curve (*b*) were computed by Eq. 71. The ordinates of curve (*c*) were computed by Eq. 74. There is fairly close agreement between curves (*a*) and (*b*), indicating that the error

[32] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York and London, 1936, Chapter V, p. 259.

introduced by assuming pure bending is not great. Curves (*b*) and (*c*) agree closely at the higher ratios of l/b, but curve (*c*) is considerably below curve (*b*) at the lower ratios of l/b.

Applying Eq. 71 to the solution of the 36WF160 beam, the following comparisons may be made:

$$\frac{l}{b} = 20; \qquad f = 3{,}331{,}100$$

$$\times \sqrt{0.00005854 + 0.00019994} = 53{,}500$$

$$\frac{l}{b} = 100; \qquad f = 133{,}250$$

$$\times \sqrt{0.00146344 + 0.00019994} = 5{,}430$$

For $l/b = 20$, the value of the second term under the radical is more than three times that of the first term, whereas for $l/b = 100$, the value of the second term is less than one seventh that of the first. Obviously for short beams (low values of l/b), the second term may not be ignored, as was done in Eqs. 72 and 74.

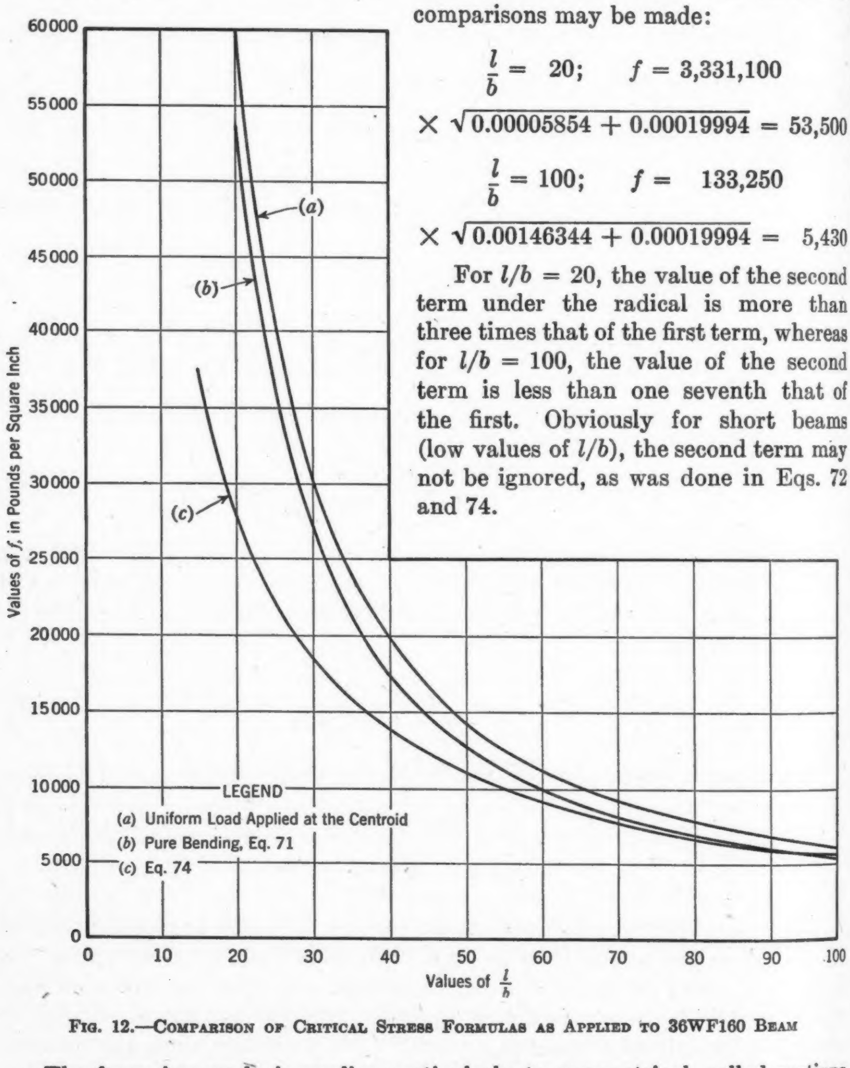


FIG. 12.—COMPARISON OF CRITICAL STRESS FORMULAS AS APPLIED TO 36WF160 BEAM

The foregoing analysis applies particularly to symmetrical, rolled sections consisting of one web plate and two flanges. For built-up girders, consisting of flange angles, cover plates, and web plate, the approximations of Eq. 73 are no longer valid, since a large part of the flange area is concentrated in the flange angles. For this reason, the de Vries formulas do not accurately reflect the effect on the critical stresses of making minor changes in the cross section

of a girder. For example, in Figs. 11(*a*) and 11(*b*), by comparing the curves ABCD, the reduction in critical stress due to the omission of the two cover plates is seen to vary between 20% and 25%. On the other hand, curves AEF, representing the de Vries formulas, indicate a constant reduction of 37.5% for all values of l/b greater than 20.

The formulas proposed in the paper are particularly applicable to rolled beams with high ratios of unsupported flange length to flange width, but they should not be applied to beams with low ratios of unsupported flange length to flange width.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

MISSISSIPPI RIVER CUTOFFS

Discussion

By W. E. Elam, C. L. Hall, H. D. Vogel, and Harry N. Pharr

W. E. Elam,[4] M. ASCE.[4a]—As complete a summary of the Mississippi River cutoffs as could be desired is contained in this paper. However, the political situation involved was not revealed as much as merited.

Two great needs brought about this development in flood control plans. Protection from floods, beginning at New Orleans, La., about 1717 and gradually working northward, was the first factor and was undertaken solely by state and local interests. Flood control remained a local responsibility until about 1917, when the first Flood Control Act was passed by Congress and the federal government accepted about 50% of the burden of levee construction. The great flood of 1927 brought the realization that flood control was a national problem and responsibility.

Navigation is the second factor and had its inception in 1879, when the Mississippi River Commission was set up by Congress to foster and develop navigation. Flood control was not considered to be a function of the Commission. Local interests were left with this part of the problem; they willingly accepted the responsibility until recurring disasters demonstrated that the federal government had to solve the problem or this great alluvial valley would be destroyed as a national asset.

The effort by two agencies, with different objectives, resulted in confusion and failure to attain flood protection. Navigation interests caused the Mississippi River Commission to be established and probably were opposed to cutoffs, believing that lengthening the river would produce a sluggish river, which would best serve the needs of navigation. This could have been the reason the Commission adopted the policy of preventing cutoffs. Reports and papers supporting this policy have added confusion to an already complicated problem. Even the official reports after the 1927 flood specifically rejected cutoffs as aids to flood control. Local authorities followed the lead of the official reports and also condemned cutoffs. Too much credit cannot be given Maj.-Gen. Lytle Brown, M. ASCE, formerly Chief of Engineers, and Maj.-Gen.

Note—This paper by Gerard H. Matthes was published in January, 1947, *Proceedings*.

[4] Chf. Engr., Mississippi Levee Dist., Greenville, Miss.

[4a] Received February 1, 1947.

H. B. Ferguson, M. ASCE, formerly President of the Mississippi River Commission, for adding cutoffs to the flood control plan when the entire valley officially opposed their use. This one feature made flood control possible.

Only one small part of the problem was covered in the discussion of the Greenville Bends[2] mentioned by Mr. Matthes. However, the conclusions in that discussion broadened the scope of the paper considerably. An article in *Engineering News-Record*[5] went into the matter more fully and considered the problem of the entire river from Cairo, Ill., to the Gulf of Mexico. This same subject is also covered in a book.[6]

The conclusion that the river tends to return to a certain length may not be strictly true. The function of the river is to carry water to the sea. The Mississippi has always brought along soil, sand, and gravel which fills the alluvial valley to a gradual slope from the upper end to the Gulf of Mexico. This material is shoved out of the main channel or deposited in slack water to form bars, generally on the concave side of bends. Bars thus formed on the points cause the caving of opposite banks. Some of these caving bends above and below a point tend to approach each other and eventually to produce a natural cutoff. As Mr. Matthes has stated, some of these proved disastrous. However, if these same cutoffs had been laid out intelligently, they could have been beneficial.

There is the possibility that man could well select a length of river that would produce a maximum flood control benefit with a minimum interference with navigation. Data to do this are probably in existence. Recognition should more often be made of the facts that the Mississippi River carries water, silt, sand, and gravel and that all these move irresistibly together and all must be taken into account if the caving bank situation is to be mastered. With the mass of reports and writings about the Mississippi, framed about the old theory that cutoffs would surely breed disaster, it is little wonder that much confusion still exists and progress is being halted by old theories based on erroneous assumptions. The Army Engineers and the Mississippi River Commission probably will eventually clarify this situation since these groups seem best qualified to work out a balanced long time program. They are on the right track, but many details yet remain to be fitted into the picture.

Mr. Matthes' estimate of the lowering of flood stages due to the cutoffs is about as accurate as any. There is no way to estimate the effect accurately. Flow conditions have been so altered that even the rates of runoff and of discharge are not comparable. What many engineers call "valley storage" is a part of the discharge area where the velocity is checked in the wide overbank channel. Valley storage is beneficial. Storage in the backwater areas is detrimental. The vast volume of water withdrawn into these areas in advance of the maximum discharge merely served to hold the river stationary at these points longer. If another flood formed in the valley in a week or ten days, as frequently happened, higher stages resulted than would have if the water had been kept out of the backwater areas.

---

[2] *Transactions*, ASCE, Vol. 93, 1929, p. 937.

[5] "Flood Control Through Slope Correction," by W. E. Elam, *Engineering News-Record*, June 28 1928, p. 996.

[6] "Speeding Floods to the Sea," by W. E. Elam, Hobson Book Press, New York, N. Y., 1946.

The river seems to have done a much better job than expected in transporting and moving the detritus out of the channel. It is possible that the increase in velocity kept much of this material from dropping in unwanted places. It is noted that, in certain reaches, such as Sessions, or Francis (mile 604 to 612 "Above Heads of Passes"), there is a tendency for an excess of detritus to clog the channel and to cause excessive caving. Probably the best treatment for this condition would be to shorten the river, where possible, at these points. Thus, the carrying capacity of the river should become more uniform and at the same time the old channel should absorb a lot of the excess detritus. It may be that more shortening has been effected than is necessary; but, as Mr. Matthes states, gradual lengthening rapidly will compensate for this condition. It is far better this way.

C. L. Hall,[7] M. ASCE.[7a]—The reasons that led to creation of artificial cutoffs on the Lower Mississippi River are clearly stated in this paper, which also well describes the methods of accomplishing these channel changes. There seems to be nothing to criticize in the paper—as far as it goes.

Mr. Matthes states (see heading, "River Steepening") that "it became clear that the river should not be straightened unduly" and also that "a 'braided' river would have impaired navigability." Cutoffs are essentially instruments for straightening the river, although perhaps not unduly. Obviously, however, the tendency of the actions taken is toward "braiding" and the consequent impairment to navigability must be resisted by increased maintenance. Since the beneficial effect of the reduction of floods caused by the cutoffs is purely economic in character, there is no way of determining the net benefit except by comparing the value of the flood reduction secured with the additional costs of maintenance. Any flood control works made unnecessary, or any maintenance costs reduced in amount by the new scheme, constitute, of course, part of the value of flood reduction.

A judgment on the utility of the plan therefore requires answers to three questions:

(1) How much are costs of channel maintenance increased by cutoffs?
(2) What flood control works are made unnecessary by the plan?
(3) To what extent is levee maintenance reduced?

There is nothing in the paper to show that these questions have even been propounded: There is certainly no sign of an answer to them. Until the questions are answered, it is probable that the members of the profession will reserve opinion on the question of cutoffs.

H. D. Vogel,[8] M. ASCE.[8a]—Several reasons have been outlined for the earlier, uncompromising attitude of the Mississippi River Commission with respect to cutoffs. Principal among these was the fear engendered by observations on cutoffs that had occurred naturally during high water periods. How-

[7] Col., Corps of Engrs., Beach Erosion Board, Washington, D. C.

[7a] Received February 3, 1946.

[8] Col., Corps of Engrs., U. S. Army, Washington, D.C.

[8a] Received February 13, 1947.

ever, it should also be recalled that, until the Flood Control Act of 1928, funds appropriated by Congress were too limited and restricted as to use to do the research and the investigational and experimental work necessary to find a solution to the problem. The only money available in any amount was that appropriated for the benefit of navigation; nevertheless, the existing levees had to be protected against incursions by the river at its bends. Revetments could be placed as aids to navigation; hence, revetments were placed and bends were held.

Plans for the United States Waterways Experiment Station were made in the summer of 1929 at Memphis, Tenn. By December of that year it was decided to build the station in Vicksburg, Miss. Early in December, as director-elect of the station, the writer visited Yucatan Point with the late Gen. T. H. Jackson, M. ASCE, then President of the Mississippi River Commission, who decided to desist from attempts to prevent the upper bend of the Mississippi River from cutting through into the channel of the Big Black River. Such a cutoff would develop slowly and would not endanger any levees. By permitting the break-through to occur, the Big Black River would act as a natural pilot channel across the neck of Yucatan Point to the lower bend. The cutoff of the Mississippi River into the Big Black River occured between December 25, 1929, and January 1, 1930. This, the first cutoff on the Mississippi River deliberately made by man (for the decision to withhold action against it constituted action for it), occurred quietly and without attendant catastrophe.

Encouraged by observations on the Yucatan Cutoff, General Jackson ordered that a model of the Greenville Bends be built and tested by the U. S. Experiment Station as one of its first two studies. The Greenville Bends model was built during the summer of 1930 and operated throughout the succeeding fall and winter.

Results of tests with this model are indicated in the following excerpt from an article in *Engineering News-Record:*[9]

> "The [Greenville Bends] model showed a maximum lowering of 2.2 ft. due to a cutoff at Tarpley Neck, the influence being felt for a distance of 45 miles above with no change below. This was in almost direct contradiction to the old theory, offered first in this country by Humphreys and Abbott, that stages are increased below a cutoff by the same degree they are decreased above.
>
> "The model gave no indication of detrimental effects due to cutoff, but on the other hand showed a slight tendency toward improved conditions at Leland Neck. The reduced head above Leland Dike, produced by a cutoff at Tarpley, relieved the strain on that structure to some degree and resulted in lowered velocities through it. It is to be expected that Chicot Point will will be severely attacked by increased currents through the cutoff; but if erosion occurs, it will be along the line of a shallow chute now existent, and revetment of the east bank will resist attack. Erosional adjustments that follow will produce a sinuous channel in due course of time."

Encouraged by results of the experiments with models of the Greenville Bends, General Jackson began plans for an artificial cutoff across Diamond

[9] "Application of Model Research to Mississippi Flood Problems," by Herbert D. Vogel, *Engineering News-Record*, July 16, 1931, p. 87.

Point; and comprehensive model tests of an extensive river reach were ordered in the spring of 1932. Actual field work on the cutoff was begun during the following summer, shortly before General Jackson was relieved by Gen. Harley B. Ferguson, M. ASCE, as President of the Mississippi River Commission. General Ferguson pushed the work on Diamond Point with such energy that the final break-through was effected on January 8, 1933.

General Ferguson, also, immediately upon taking office as President of the Mississippi River Commission, ordered comprehensive studies directed toward the creation of a whole series of cutoffs throughout the length of the lower river. Mr. Matthes, who came to Vicksburg with General Ferguson, has told the story from this point.

In commenting upon the work begun by General Jackson, there is no desire or intent to detract from the accomplishments of General Ferguson. The final result—control of the Mississippi—is important enough to provide undying glory for both. General Jackson pioneered the way, overcoming prejudices and fears of long standing; General Ferguson, with indomitable courage, applied the lessons developed by General Jackson's tests to the river as a whole. Mr. Matthes, in characteristic fashion, has declined to describe his own role in this great engineering development. It was he, however, who stood always beside General Ferguson when he was criticized for going too far and too fast. It was he who interpreted many of the laboratory results and put them to practical application. His present paper comprises a valuable contribution both to engineering knowledge and to the Mississippi River history.

HARRY N. PHARR,[10] M. ASCE.[10a]—A statement of the purpose and a clear description of the effect of the cutoffs made on the Mississippi River are presented in this paper.

The conception that the effect of a cutoff is to raise the stage downstream arises from the fact that, as the stage rises in a stream of varying slopes, the water surface tends toward a uniform slope. Until the cutoff in its process of development regulates the disarranged alinement, slope, and section of the stream channel, the velocity of the current immediately below the cutoff is somewhat retarded, the impetus of the stream is diminished, and the tendency there is to increase the stage. This effect extends a short distance downstream to about the point where the regular alinement, slope, and section of the stream are reached. In a series of cutoffs within the influence of each other this result is modified except at the cutoff farthest downstream. The effect of the decrease in valley storage, because of stage reduction resulting from the cutoff, on the increase in stream discharge below the cutoff diminishes with the usual occurrence of slight daily rises and increase in duration of stage near the flood crest.

Immediately above its location the effect of a cutoff is to increase the velocity of the stream, to lower the stage, and, in some situations, to increase the tendency to cave the concave bank. Below bankfull stage the action of drawing the current away from the concave bank and across the toe of the opposite point

[10] Civ. Engr., Member, Mississippi River Comm., War Dept., Memphis, Tenn.

[10a] Received February 18, 1947.

bar is not effective in some situations beyond the influence of the cutoff, as at Hardin Cutoff (676 Above Heads of Passes), made in 1942. In Walnut Bend, the accelerated rate of caving along the concave bank necessitated installation of bank revetment, as the current was not materially deflected across the opposite point bar despite occasional dredging in an effort to assist in accomplishing that result.

James B. Miles was among the early advocates of cutoffs to reduce flood heights in the Lower Mississippi River. Following the disastrous flood of 1897, he proposed a plan to the Nelson Subcommittee of the Committee on Commerce of the United States Senate which made an examination, inquiry, and report on the problems of flood control and improvement of the Mississippi River.[11] Mr. Miles stated:

> "I find there are 26 places at which cutoffs could be made. Of these about 16 are such as could be made at but little cost or time * * *. The distance across these necks or narrow places is from one to three miles, except one of five miles. The cost would be in digging a ditch or canal across these necks. Some would require large and some small ditches * * *. The ditch or canal should be dug so as to turn the current of the river above as directly as possible into the channel or current below. About all the cutoffs have been made where the neck was narrowest. This, in most instances, has thrown the current against the bank on the opposite side so as to cause great destruction. * * * I do not claim that the river can be made straight but I claim that the distance can be reduced * * *. The distance from Cairo to the Gulf could be shortened by cutoffs some 170 miles. * * * The cutoffs between Arkansas River (mouth) and Greenville should lower the water at the Arkansas nearly or quite 10 feet."

The plan which Mr. Miles advocated is somewhat similar to the plan executed. He was not a professional engineer, but he was a student of the river. It is remarkable that, 50 years ago, he selected the exact number of cutoffs and estimated both the exact mileage reduction in length of river and about the amount of reduction in stage as that of the plan executed.

To the reasons given by the author as to why natural cutoffs were not permitted to occur during the period from 1884 to 1929 should be added the fact that, previous to the first Flood Control Act of 1917, the appropriations by Congress of funds for the improvement of the Mississippi River were generally restricted to works in aid of navigation; works solely for the purpose of flood protection were prohibited. Levees had been constructed as auxiliaries to the plan of channel improvement. This resulted in the adoption of a policy of stabilization rather than a change of channel location, and any benefit to flood control resulting from works designed to aid navigation was considered incidental. Since 1917 Congress has authorized works in aid of flood control as well as works for the improvement of navigation. The principal objective of the cutoffs is the control of floods and any resulting improvement aiding navigation is usually incidental.

[11] *Senate Report No. 1433*, 55th Cong., 3d Session, December 15, 1898, pp. 296–301.

Corrections for *Transactions:* In January, 1947, *Proceedings,* in Table 2, page 13, replace the column headed "Cutoffs" by a column headed "Lengthening," with the following values:

Interval	Lengthening
55	7
62	11
34	6.8
13	0.7
16	2.7

On page 13, in line 16, change "new river" to "net river"; in Table 4, footnote [a] capitalize "Head of Passes"; and, on page 17, change line 26 to read: "* * * at upstream cutoffs and at other rectification operations appears to have affected." In Fig. 1 on page 4, multiply the scale of miles by ten.